

Essentials of Constrained Optimization

Economic optimization problems generally involve individuals making choices in a setting of facing individuals and firms are constrained in some manner so that the above "unconstrained" maximizing (or minimizing) techniques *can not* be directly applied by either the individuals themselves, or those attempting to model their behavior. (For example, utility maximizing individuals are constrained by a budget constraint.) There are two widely used methods for solving constrained optimization problems.

I. The Substitution Method

- A.** The Substitution Method. In some cases, it is possible to "substitute" the constraint into the objective function (the function being maximized) to create a new composite function that fully reflects the effect of the constraint. *I will use this method extensively through out this course.*
- For example consider the separable utility function: $U = x^5 + y^5$ to be maximized subject to the budget constraint $100 = 10x + 5y$. (Good x costs 10 \$/unit and good y costs 5 \$/unit. The consumer has 100 dollars to spend.)
 - Notice that, from the constraint we can write y as $y = [100 - 10x]/5 = 20 - 2x$
 - Substituting into the objective function (in this case, a utility function) yields a new function entirely in terms of x: $U = x^5 + (20 - 2x)^5$
 - This new function accounts for the fact that every time one purchases a unit of x one has to reduce his consumption of y. (How?) Moreover it is a function with just one control variable, x.
 - Differentiating with respect to x allows the utility maximizing quantity of x to be characterized:

$$d[x^5 + (20 - 2x)^5] / dx = .5 x^{-.5} + .5(20-2x)^{-.5}(-2) = 0 \text{ (at U max)}$$

- The derivative will have the value zero at the constrained utility maximum. Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for x yields:

$$4x = 20 - 2x \Rightarrow 6x = 20 \Rightarrow x^* = 3.33$$

- Substituting x^* back into the budget constraint yields a value for y^*

$$y = 20 - 2(3.33) \Rightarrow y^* = 13.33$$

- No other point on the budget constraint can generate a higher utility level than that $(x^*, y^*) = (3.33, 13.33)$.

II. Lagrangian Method

- A.** The Lagrangian method uses a somewhat different method of modifying the objective function being maximized to account for (equality) constraints that restrict the feasible range of the objective function. The method operates as follows:
- Transform all of the constraints into a form that equals zero. For example, given an algebraic expression, like a budget constraint, $W = P_1X_1 + P_2X_2$, becomes a form like: $W - P_1X_1 - P_2X_2 = 0$.

- Create a new objective function, called a *Lagrangian*, by multiplying each constraint (in its zero form) by a Lagrangian multiplier λ and adding the result(s) to the objective function. For example, if the utility function is $U = u(X_1, X_2)$ and the constraint is as above the Lagrangian would be:

$$L = u(X_1, X_2) + \lambda(W - P_1X_1 - P_2X_2)$$

- Differentiate the Lagrangian with respect to all the *control variables* and the Lagrangian multipliers. (In the example, the control variables are X_1, X_2 and λ . The consumer does not control the other parameters of the optimization problem. The prices of the goods, and wealth are generally assumed to be exogenous at the level of the consumer.)
 - $L_{X_1} = U_{X_1} - \lambda P_1$
 - $L_{X_2} = U_{X_2} - \lambda P_2$
 - $L_{\lambda} = W - P_1X_1 - P_2X_2$
 - (Subscripted variable names denote partial derivatives with respect to the variable subscripted. $L_{X_1} = \partial L / \partial X_1$, $U_{X_1} = \partial U / \partial X_1$, etc.)
- At the constrained maximum (or minimum) all of these partial derivatives will equal zero. These first order conditions can be algebraically manipulated to deduce properties of the constrained optimum.
 - For example, one can use the above to show that $U_{X_1} / U_{X_2} = P_1 / P_2$.
 - That is to say, at the utility maximum, the marginal rate of substitution between X_1 and X_2 equals their relative price.

III. The Implicit Function Theorem

- A.** In most cases, the first order conditions of an "abstract" optimization problem completely characterizes the optimum values of all control variables. However, it is often difficult to directly interpret "the solution" in an intuitive and meaningful way.
- B.** Algebraic manipulation of the first order conditions can often shed a bit of light on the properties of the solution characterized.
- For example, the first order conditions of a consumers constrained utility maximizing choice can generally be manipulated to show that the marginal rate of substitution between any two goods equals their relative prices.
 - (This is the algebraic method of showing that the constrained utility maximum lies at a point where an indifference curve is tangent to the budget constraint.)
- C.** The implicit function theorem allows additional properties to be deduced from the first order conditions. The implicit function theorem allows the first order conditions to be used:
- to characterize the solution (optimal value of the control variable(s)) as a function of other parameters of the optimization problem.
 - and to characterize the comparative statics of the solution.
 - To do the latter, one applies the "implicit function differentiation rule" to find derivatives of the "ideal values" of the control variables as parameters of the choice problem change.

Mathematical Appendix for Lecture 1

- b. For example, in a consumer choice problem one can determine rate at which the optimal quantity of a good changes as the consumer's income changes or as some price changes. ($dQ^*/dY, dQ^*/dP_s, \dots$)

D. The *Implicit Function Theorem* (see Chiang 205 - 206): Given a function such that:

- i. $F(Y, X_1, X_2, \dots, X_m) = 0$
- ii. Where the function F has continuous partial derivatives $F_Y, F_{X_1}, F_{X_2}, \dots, F_{X_m}$ and at point $(Y^0, X_1^0, X_2^0, X_3^0, \dots, X_m^0)$ satisfying condition i, F_Y is nonzero, *Then* there exists an m-dimensional neighborhood of the point $(Y^0, X_1^0, X_2^0, X_3^0, \dots, X_m^0)$, N, in which an *implicit function* exists that characterizes each of the "ideal" values of the control variables of F as a function of non-control variables (the "parameters" of the choice problem): $Y = f(X_1, X_2, X_3, \dots, X_m)$.
- iii. This function satisfies $Y^0 = f(X_1^0, X_2^0, X_3^0, \dots, X_m^0)$ in particular, and more generally, $Y = f(X_1, X_2, X_3, \dots, X_m)$ for all points within the neighborhood.
- iv. This gives the function the status of an identity within neighborhood N. Moreover, the implicit function, f, is continuous and has continuous partial derivatives with respect to X_1, X_2, \dots, X_m .
- v. The N-equation version of the implicit function theorem is broader in scope but essentially similar. (See Chiang 210 - 211.)

E. In economic applications, the original function, F, is usually the first order condition of some optimization problem, and the implicit function is the individual's (or firm's) demand (or supply) function.

F. The *implicit function differentiation rule* is used to characterize the partial derivatives of the implicit function "generated" as above.

- i. The rule is surprisingly simple for single equations. The partial derivative of implicit function Y with respect to X_i is simply: $Y_{X_i} = F_{X_i} / -F_Y$ (where subscripts denote partial derivatives with respect to the variable subscripted).
- ii. The derivation of this rule is based on the definition of F and the use of *total derivatives*.
 - a. Recall that at the point of interest, $F(Y^0, X_1^0, X_2^0, X_3^0, \dots, X_m^0) = 0$
 - b. So the total derivative of F has to add up to zero:

$$dY F_Y + dX_1 F_{X_1} + \dots + dX_m F_{X_m} = 0$$
 - c. Consequently, if we allow only X_i and Y to vary: $dY F_Y + dX_i F_{X_i} = 0$
 - d. Solving this expression for dY/dX_i yields: $dY/dX_i = F_{X_i} / -F_Y$
- iii. The implicit function differentiation rule allows one to characterize how an optimal value varies as parameters of the decision problem vary.

G. Example: Determine the slope of an individual's demand function

- i. Suppose that "Al" has a utility function, $U = u(X_1, X_2)$ which is monotone increasing, twice differentiable and strictly concave.
- ii. Al wants to find the utility maximizing combination of X_1 and X_2 given the budget constraint that he faces, $W = P_1 X_1 + P_2 X_2$.
- iii. Using the substitution method we write U as: $U = u(X_1, (W - P_1 X_1)/P_2)$
- iv. Differentiating with respect to X_1 yields f. o. c.: $U_{X_1} + U_{X_2} (-P_1/P_2) = 0$
- v. Any value of X_1 that satisfies this first order condition will maximize utility. Denote such an X_1 as X_1^*
- vi. At X_1^* , the first order condition is a function like F in the definition of the implicit function theorem. Since the first order condition is differentiable (remember that we assumed that U was twice differentiable), an implicit function exists which characterizes X_1^* as a function of the other parameters of the choice problem. $X_1^* = x(W, P_1, P_2)$
- vii. This function is Al's *demand* function for X_1 .
- viii. The sensitivity of Al's demand for a change in the price of good 1 can be computed using the implicit function differentiation rule:
 - a. $X_1^* P_1 = F_{P_1} / -F_{X_1}$
 - b. which from the first order condition (the function F for this problem) can be written as:

$$\frac{[U_{X_1 X_2} (-X_1/P_2) + U_{X_2} (-1/P_2) - U_{X_2 X_2} (P_1/P_2)(-X_1/P_2)]}{-[U_{X_1 X_1} + 2 U_{X_1 X_2} (-P_1/P_2) + U_{X_2 X_2} (-P_1/P_2)^2]}$$
- ix. The "sign" of this derivative determines whether Al's demand curve slopes down or not. The "sign" can be determined by carefully attempting to determine whether each of the partial derivatives in the expression is positive or negative.
 - a. From the original characterization of U we know that all of the first partial derivatives are positive and all of the second derivatives are negative (why?).
 - b. If we also assume that all of the cross partials are positive, Al's demand curve is downward sloping, $X_1^* P_1 < 0$.