I. Review: Where We Are.

A. During the first lecture, we explored the meaning of rational choice and began to discuss how a rational choice model of human action could be constructed.

- The simplest model of rational choice assumes that individuals have consistent preferences, that is to say transitive preferences that can be represented with a (ordinal) **utility** function.
- Somewhat richer definitions might also argue that rational decision makers understand the consequences of their decisions so that they always choose the **best** means to advance their ends--at least given the information available to them.

B. During the last three weeks, we have used the rational choice model (in its "utility maximizing" sense) to model human decisions in a setting where one person's welfare (utility) depends in part on the decisions of other players.

C. Such decision environments are the main domain of Game Theory.

D. We began with the simplest possible game setting, in which there were two person and two strategies, and explored how different payoff structures affect the equilibrium outcomes of a variety of such games.

i. For example, we demonstrated that the PD-type of payoff structure implies the existence of pure dominant strategies, and that the typical Nash equilibrium is not Pareto optimal.

ii. We also noted that suboptimal equilibria may also arise in assurance games.

iii. In general, whether the Nash equilibrium outcome of a particular game is Pareto Optimal or not depends on the relative sizes of the payoffs associated with each players strategies.

iv. These are partly determined by nature of the game (which is characterized by strategy sets and payoff functions) and partly by the strategy choices of the other players.

E. We then showed how the 2x2 game matrix can be generalized to analyze settings in which there are a more than two strategies, and more than two players.

i. For example, we extended the 2x2 game to three strategies, and then to a continuum of strategies (in the lottery game).

ii. We used the lottery game to illustrate how decisions can be made in a setting in which there are an infinite (continuum) of strategies.

- To find the equilibrium in such games, we first found the **best reply functions** of the individual players.
- Then we found the combination of strategies that occurred if all players were simultaneously on their best reply functions.
- This combination of strategies was a Nash equilibrium of the game, because the fact that every player was simultaneously on their best reply function implied that each had maximized his utility (payoff)--given the choices of the other players.

iii. We also used the lottery game to illustrate how games in which a various numbers of players (N players) may participate in the game, which allowed us to characterize an infinite strategy and infinite player game.

F. For the next two weeks, we will focus on applications of these basic "one shot" game techniques. After that, we will take up another generalization of these game theoretic models, namely to settings in which there are sequences of choices and information constraints.

II. An Application from Economics: Cournot Duopoly

A. There are three widely used models of duopoly: (1) *Cournot* (based on symmetric quantity competition), (2) *Bertrand* (based on symmetric price competition), and (3) *Stackelberg* (based on asymmetric quantity competition with a first and second mover).

i. The first and last of these models is the most broadly used.

ii. The Bertrand, however, should not be totally neglected, because the model yields simple and direct predictions about pricing and output.

B. In the **Cournot** model (some times called Cournot / Nash duopoly), two firms produce identical goods and make their *output* decisions independently of one another. Each takes the other's output given, and selects its own best output given that assumption given the *downward sloping market demand curve* for the product in question. Total output and market price are represented as equilibria to the "noncooperative" production game between the two firms.

i. As an illustration, suppose that the market in question has a demand curve: Q = 1000 - 10P.

- Assume also that firms are profit maximizers and that profit is simply revenues from sales less the cost of producing the goods sold; thus, firm "A's" profit can be written as $\Pi^A = PQ^A - C^A$.
- In order to know (or estimate) their profits they will have to know price, but market price depends on the total amount brought to the market by all firms, not simply that brought by firm "A."
- Demand curves "slope downward," which means that the more output is "brought to the market" the lower prices tend to be.

ii. Given our assumed demand curve, Q = 1000 - 10P, and the affect of total market output on market price can be written as: P = 100 - 0.10*Q.

- (This way of writing a demand curve is often called an inverse demand curve.)

iii. If there are just two firms, A and B, then $Q = Q^A + Q^B$ and firm A profits can be written as:

- $\Pi^A = [\ 100 - 0.10*(Q^A + Q^B)\ ]\ Q^A - C^A$

iv. To simplify a bit more, let us also assume that the cost function is the same for each firm and equals:

- C = 5Q
- so the profit of firm A is simply $\Pi^A = [\ 100 - 0.10*(Q^A + Q^B)\ ]\ Q^A - 5Q^A$
- or (multiplying), $\Pi^A = 100 - 0.10(Q^A)^2 - 0.10Q^AQ^B - 5Q^A$
- Notice that firm A's profit (his payoff in this game) is affected by the other player's output decision, as is typical in a game setting.

v. To find firm A's profit maximizing output, we need to find the "top" of the profit function, which can be found where the slope of the profit function is zero.

- Differentiating with respect to Q^a and setting the result equal to zero yields:

$$100 - 0.20\ Q^A - 0.10Q^B - 5 = 0$$

vi. Solving this "first order condition" for Q^A allows us to characterize the output level that maximizes firm A's profit for each output level that B might choose:

- $100 - 10Q^B - 5 = 0.20\ Q^A$
- $95 - 0.10Q^B = 0.20\ Q^A$
- $475 - 0.50Q^B = Q^A$
- or $Q^A = 475 - 0.50Q^B$

- (This equation is the best reply function of firm A. It tells firm A how much to produce to maximize his own profits for any output level of firm B.)

vii. A similar best reply function can be found for firm B (Do this as an exercise!)

- $Q^B = 475 - 0.50Q^A$

viii. At the Nash equilibrium, both firms will be on their best reply function--that is to say both firms will be maximizing their profits given the other firm's output.

- This implies that
- $Q^B = 475 - 0.50Q^A$ and
- $Q^A = 475 - 0.50Q^B$
- will simultaneously satisfied.

ix. Substituting for Q^B into Q^A allows us to find the Nash equilibrium levels of output for firm A. allows us

- $Q^{A*} = 475 - 0.50Q^{B*}$ which implies that $Q^A = 475 - 0.50 (475 - 0.50Q^A)$
- multiplying through yields $Q^A = 475 - 475/2 + 0.25Q^A$
- gathering terms $0.75 Q^A = 475/2$
- or that $Q^{A*} = (4/3)(475/2) = (950/3) = 316.6$
-
- By symmetry (show this as an exercise) we also know that $Q^{B*} = 316.6$,
- which implies that total output, $Q^* = 633.3$, namely, the sum of the outputs of the two firms.

x. Substituting back into the profit function allows us now to determine each firm's payoff:

- the profit of firm A is simply $\Pi^A = [100 - 0.10*(Q^{A*} + Q^{B*})] Q^{A*} - 5Q^{A*}$
- or, substituting for the "Qs" we have $\Pi^A = [100 - 0.10*(633.3)] 316.6 - 5(316.6)$
- $\Pi^A = (95)(316.6) - 63.3 (316.6)$ or $(31.3) 316.6$

C. In equilibrium both firms are simultaneously maximizing profits

- Geometrically this occurs at the intersection of the two firm's best reply curves.

i. As an exercise, show that total output increases relative to the monopoly market and prices fall.

- Recall that a monopolist will simply maximize profit, which in this case is:
- $\Pi = [100 - 0.10*(Q)] Q - 5Q$
- Find the profit maximizing output for the monopolist by differentiating its profit function with respect to Q, setting the result equal to zero, and solving for Q.
-
- Next find the profit levels of the two firm Cournot Duopoly).
- Which industry earns higher profits?
- (Hint: the result should be 475 which, of course is less than 633.3.)
- In which case are consumers better off?

III. The Bertrand duopoly model

A. The Bertrand duopoly model assumes that each firm **competes on price** rather than output. In effect, each firm bids for the business of consumers in a market with homogeneous products by choosing a price for its products.

- If consumers have complete information, each firm can secure essentially the entire market by charging a price just below that of its competitor.
- A sequence of profit maximizing prices offers, thus, declines, until at the Nash equilibrium, neither firm earns a positive economic profit.
- In the case where each produces via constant returns to scale, this implies that the prevailing market *price* **under this form of duopoly** *is exactly the same as that of a perfectly competitive* market.

IV. The Stackelberg Duopoly Model

A. In the **Stackelberg** model, a market is shared by two firms. However, in contrast to the Cournot model, there is a "natural" order to the game.

i. One firm announces its output first (the leader) and the other announces his second (the follower).

- You can think of the Cournot example as representing the decisions of two farmers who independently bringing baskets of potatoes to a local market, without knowing exactly what the other does.
- At the Cournot-Nash equilibrium, each is satisfied with the quantity of potatoes that they brought to market given what the other farmer has done.

ii. In the Stackelberg case you can imagine farmer B bringing his potatoes to market after observing the quantity brought by farmer A.

- (The Stackelberg case makes the sequence of decision making important for the first time--and, thus represents a generalization of our "one shot" games.)

iii. The Stackelberg type of model can be used to represent a wide variety of choice settings in which there is a "natural" first and second mover, as we will see later in the course.

B. Rationality implies that the first mover (leader) will tries to take account of the effects of his output decision on the follower, because their joint output determines the market price of the good sold in the market (as in the Cournot duopoly case).

i. In effect it chooses its profit maximizing output given the "profit maximizing output schedule" of its competitor.

ii. A typical Stackelberg model: Let "a" be the leader.

iii. The definition of profit has not changed, so: $\Pi^A = PQ^A - C^A$

- If we use the same specific functional forms that we used in the Cournot Duopoly problem we will have:
- $\Pi^A = [100 - 0.10*(Q^A + Q^B)] Q^A - C^A$

iv. However, in this case, A does not take Q^B as given, but knows that B's behavior will be affected by his own decision about output.

- In fact, A knows that B will choose the quantity that maximizes his own profits given the quantity brought to market by A.

v. From our previous effort on the Cournot duopoly model, we know (and so does A) that B will follow the rule: $Q^B = 475 - 0.50Q^A$, as per B's best reply function.

vi. Thus, A anticipated B's response when bringing his own goods to market, and maximizes:

- $\Pi^A = [100 - 0.10*(Q^A + Q^B)] Q^A - C^A$ where $Q^B = 475 - 0.50Q^A$ and $C^A = 5Q^A$, thus,
- $\Pi^A = [100 - 0.10*(Q^A + (475 - 0.50Q^A))] Q^A - 5Q^A = 100Q^A - 0.10(Q^A)^2 - 47.5Q^A + 0.05(Q^A)^2 - 5 Q^A$

vii. In order to find the profit maximizing output, we again differentiate Π^A with respect to Q^A and set the result equal to zero:

- $52.5 - 0.10\, Q^A - 5 = 0$ (in economic terms this is MR - MC = 0)
- or $Q^A = 475$
- Note that the marginal revenue function now includes the effect of the leader's anticipated effect on firm b's change in output and on the prevailing price, as well as the leader's own effect on price.

viii. To find out total market supply we now go back to firm B's best reply function and substitute for Q^A:

- $Q^B = 475 - 0.50Q^A = 475 - 0.50\,(475) = 237.5$
- So, the total output is now 237.5 + 475 = 612.5 which is between the monopoly output and the Cournot output.
- So profits will be higher under Stackelberg than under Cournot, and they will be higher for the first mover than for the second mover.
- (Calculate the profit levels for A and B as an exercise.)

C. The Stackelberg type of model can be used to model a wide range of settings in which there is a "natural order" to the decisions of interest or in the information sets of the players of interest.

i. For example in crime control, normally the legal system "moves first" by imposing a schedule of fines and allocating resources to the police and courts. The criminal moves second by choosing his crime rate.

ii. Clearly the best strategy for the "legal system" (as desired by the median voter) is to take account of how the typical criminal will respond to its decision regarding penalties and policing effort

V. Another example from economics: an Extension and Generalization of Cournot's model of duopoly to markets with N firms: Monopolistic Competition

A. The Cournot duopoly model provides a very natural method of modeling the effects of entry. One can easily extend the model to include 3, 4, 5 ... N firms. The result will be market prices and output that more and more conform to the perfectly competitive result.

B. Consider an extension of the linear Cournot Duopoly problem in which there are N identical firms rather than just 2. Suppose each firm has cost function $C = cQ$ and let the inverse market demand be: $P = XY - eQ$ where market production is $Q = [Q^a + (N-1)Q^o]$

i. The first order condition that characterizes maximal profits for firm "a" is now

$$(XY - e(N-1)Q^o - 2eQ^a) = c$$

ii. Firm a's reaction function is thus: $Q^a = [XY - e(N-1)Q^0 - c\,]/\,2e$

iii. and the Cournot Nash equilibrium output for a typical firm at the symmetric equilibrium is : $Q^* = [XY - c] / (N+1)e$ and total output is N times as large: $Q^* = \{[XY - c] / e\}\,\{N/(N+1)\}$

iv. As N approaches infinity, the total output approaches the competitive equilibrium. *Perfect Competition, thus, is a Limiting Case of entry in a Cournot-Nash type model.*

C. In general there are a wide variety of models of imperfect competition, which vary mainly with respect to the manner in which players anticipate or fail to anticipate reactions of other players in the game. [The so-called conjectural variation.]

VI. Externalities and Pigovian Taxation

A. In cases where an externality is generated by an activity, it will often be the case that the privately optimal activity levels will ones that are **not Pareto efficient**.

B. The easiest way to demonstrate this mathematically is with a two person non cooperative game (group) illustration.

i. Suppose that Al and Bob are neighbors. Both own barbecues, and that neither enjoys the smell of smoke and such associated with the other use of their barbecue. Let us refer to Al as Mr. 1 and Bob as Mr. 2.

ii. Let $Ui = u_i(Ci, Bi, Bj)$ for each person i (here: i = 1, 2) with Ci being food cooked indoors and Bi being food cooked outdoors by i, and Bj being food cooked outdoors by the neighbor ($i \neq j$). To make the model tractable, assume that Mr. I allocates his "kitchen time" Ti between cooking and barbecuing so that Ti = Ci + Bi for all i.

iii. Mr I's barbecuing time can be determined by maximizing U subject to the time constraint. Substituting, the constraint into the objective function to eliminate Ci yields: $Ui = u_i(\,Ti - Bi, Bi, Bj)$.

iv. Differentiating with Bi yields: $Ui_{Ci}(-1) + Ui_{Bi} = 0$. Each person will use the barbecue up to the point where the marginal cost in terms of reduced satisfaction from indoor cooking equals the marginal utility of further outdoor cooking.

v. The implicit function theorem implies that $B_1^* = b_1(B_2, T_1)$. This can be interpreted as Mr I's best reply function.

vi. In a Nash game between the two neighbors, equilibrium will occur when:
$B_1^{**} = b_1(B_2^{**}, T_1)$ and $B_2^{**} = b_2(B_1^{**}, T_2)$

C. The matter of whether this Nash equilibrium is Pareto Efficient or not is intuitively fairly obvious. Since each imposes costs on the other that are neglected, odds are they wind up in a setting where both would be better off if they produced less smoke.

D. That Al or Bob could be made better off by coordinating their behavior or not can be demonstrated in a number of ways.

i. One way to determine this is to show that a general **social welfare function**. $W = w(\,U^A, U^B)$ is maximized at by the relevant choices, given the same constraints. This can be ascertained by determining whether the first order conditions for maximizing "social welfare" are the same as those which maximize individual welfare.

ii. Another method of determining this without using a social welfare function (taken from Baumol) is to consider whether one person could be made better off at the Nash equilibrium without making the other worse off.

- For example: maximize $L = u_1(T_1-B_1^*, B_1^*, B_2^*) - \lambda(\,U_2 - u_2(T_2-B_2^*, B_2^*, B_2)\,)$ by varying B_1 and B_2
- Differentiating with respect to B1 and B2, and appealing to the envelop theorem (to eliminate effects of B1* on B2* and vice versa) yields:

$$U1_{C1}(-1) + U1_{B1} - \lambda\, U2_{B1} = 0$$

and $$U1_{B2} - \lambda\,(U2_{C2}(1) + U2_{B2}) = 0$$

E. Note that these first order conditions are different than those met for either person insofar as they imply that the externality will be internalized at the margin for both parties..

VII. Electoral Competition in a Representative Democracy

A. If two candidates can choose policy positions, and voters will vote for the candidate closest to their preferred policy, it turns out that the candidate who is closest to the median voter will win the election.

i. (This "distance based" model of voter behavior is sometimes called the spatial voting model.)

ii. If the candidates may freely choose policy positions, there is a tendency for electoral competition to cause them to select essentially identical policy positions which maximize the median voter's welfare.

iii. At this Nash equilibrium, the strong form of the median voter theorem results.

B. To see this consider the case where v(G) is the distribution of voter preferences and V^0 is the median voter's ideal point, that is to say the voter whose preferences lies exactly in the middle of the distribution, with half the voters preferring larger and half smaller values of G.

i. The votes received by a candidate 1 can be written as $V_1(G_1) = \int_{(-)}^{(G1+G2)/2} v(G)\, dG$ and that of candidate 2 as $V_2(G_2) = \int_{(G1+G2)/2}^{(+)} v(G)\, dG$.

ii. Each candidate wants to adopt the platform that maximize their own votes, given that chosen by the other candidate.

iii. The optimal decision of candidate 1 can be found by differentiating his vote function with respect to G_1, which yields: $V_{G1} = v((G_1+G_2)/2)\ /2$ which is not zero except where $G_1 = G_2$.

- (The above is positive for $G_1 < G_2$, and is negative, $-v((G_1+G_2)/2)\ /2$, for $G_1 > G_2$.)

iv. This implies that a candidate should adopt the same platform as his competitor.

v. However, this is not really an equilibrium except at $G_1 = G_2 = V_0$. At every other place where the candidates adopt the same position, one of the candidates can always do better than the other (rather than accept a tie) by moving closer to the median voter.

vi. Thus, at the Nash equilibrium both candidates take the same position, the platforms maximize both candidate's votes (given what the other candidate has done), and also **maximize the welfare of the median voter.**

VIII. Using the Lottery Game to model Status Seeking and Rent Seeking

A. The rent-seeking literature has used a game theoretic frame of analysis, which like that of the Chicago models, has been more focused on interests than on elections.

i. The core rent seeking model regards the process to be analogous to a lottery.

ii. The special favors which may be obtained through government--tax breaks, protection from foreign competition, contracts at above market rates etc.-- are the prize sought by rent seekers.

iii. The process by which these prizes are awarded is considered to be complex in that a wide variety of unpredictable personalities and events may ultimately determine who gets which prize.

- None the less, it is believed that the more resources are devoted to securing preferential treatment (e. g. the better prepared and more widely heard are the "rationalizations" for special preference) the more likely it is that a particular rent-seeker will be successful.
- Contrariwise, the greater the efforts of alternative rent-seekers, the less likely a particular rent-seeker is to succeed.
- As a **first approximation** of this political influence game, investments in political influence are often modeled as if they were purchases of lottery tickets.

B. The Basic Rent-Seeking Game

i. Suppose that N risk neutral competitors participate in a rent seeking game with a fixed prize, Π.

ii. Each player may invest as much as he wishes in the political contest.

iii. The prize is awarded to the player whose name is "drawn from a barrel" containing all of the political lottery "tickets." So, the expected prize for player i is $\Pi[R_i / (R_i + R_o)]$, where R is the value of the prize, R_i is the investment in rent seeking by player i, and t_o is the investment by all other players.

iv. If the rent seeking resource, R, cost C dollars each, the number of tickets that maximizes player 1's *expected reward* for a given purchase by all other players can be determined by differentiating the expected rent $\Pi^e = \Pi[R_i / (R_i + R_o)] - CR_i$ with respect to Ri and setting the result equal to zero.

v. $\Pi\ [1 / (R_i + R_o) - R_i / (R_i + R_o)^2] - C = 0$

vi. Which implies that: $\Pi\ [R_o / (R_i + R_o)^2] - C = 0$ or $\Pi R_o / C = (R_i + R_o)^2$

vii. So player 1's best reply function is $Ri^* = -Ro \pm \sqrt{(\Pi R_o / C)}$ Of course, only the positive root will be relevant in cases where Ri has to be greater than zero.

viii. In a symmetric game, each player's best reply function will be similar, and at least one equilibrium will exist where each player engages in the same strategy.

ix. Thus, if there are N-1 other players, at the Nash equilibrium, $R_o^{**} = (N-1)R_i^{**}$. which implies that $Ri^{**} = -(N-1)R_i^{**} \pm \sqrt{(\Pi\ (N-1)R_i^{**}/C)}$.

x. which implies that $NRi^{**} = \sqrt{(\Pi\ (N-1)R_i^{**}/C)}$ or squaring both sides, dividing by Ri^{**} and N^2 and gathering terms, that: $Ri^{**} = [(N-1)/N^2]\ [\Pi/C] = [(1/N) - (1/N^2)]\ [\Pi/C]$

C. So for example with N = 2 and C = 1, $Ri^{**} = (\Pi/4)$

i. Total rent seeking effort is N times the amount that each player invests

ii. Thus in the two person unit cost case, $\mathbf{R} = \Pi/2$.

- Half of the value of the prize is consumed by the process of rent seeking. [Illustrating Figure]
- In the more general case, $\mathbf{R} = [(N-1)/N]\ [\Pi/C] = [1 - 1/N]\ [\Pi/C]$ [See our earlier results on lottery games.]

D. The effect of entry on individual and total rent seeking expenditures can be determined by inspection or by differentiation Ciii and Bx above with respect to N.

i. It is clear that individual contributions fall as the number of rent seekers increase, but also clear that the total amount of rent seeking "dissipation" increases.

ii. In the limit, as $N \Rightarrow \infty$ the total rent seeking investment approaches the level where the value of those resources, RC, equals to the entire value of the prize, note that $\mathbf{R}^{**} C = [\Pi/C]\ C = \Pi$.

iii. The effect of increases in the cost of participating in the political influence game and/or changes in the value of the regulation to the rent-seeker can also be readily determined in this game.

E. The basic lottery model can be generalized to cover cases in which the prize is **endogenous** and where the probability of securing the prize varies, and to cases where the prize is shared rather than awarded to a single "winner take all" winner.

i. For example, $R_i^e = P(R_1, R_2, \ldots R_N)\Pi_i(\mathbf{R})$ encompasses many of these features.

ii. The affects of economies of scale may also be examined in this general framework and in the earlier explicit one.

IX. Exercises and Problems

A. Use a linear version of the Stackelberg model to determine whether the leader or follower has a larger market share and profit.

B. Critique the Bertrand model. Are two firms always sufficient to generate marginal cost pricing? Why or why not?

C. Use a Cournot-type model to represent a two person rent-seeking game. Assume that two parties compete for complete control of a market by making campaign contributions to elected officials who control the monopoly license. Suppose that the probability of gaining the monopoly privilege, $P_1 = p(R_1, R_2)$, increase with ones own contribution, R_1, but falls with that of other firm's, R_2.

i. Characterize the equilibrium level of rent seeking engaged in by two firms who could realize a profit of Π dollars if they win the monopoly privilege.

D. Discuss why behavior in such games tends to be suboptimal from the point of view of the participants and society at large. Are their cases where rent-seeking efforts do not generate a dead weight loss?

E. Homework Problems

i. Use a Cournot-type model to represent a two-firm market.

ii. Use a Cournot-type model to represent political competition within an existing government.

- Assume that two parties compete for complete control of a market by making campaign contributions to elected officials who control the monopoly license.
- Suppose that the probability of gaining the monopoly privilege, $P_1 = p(R_1, R_2)$, increase with ones own contribution, R_1, but falls with that of other firm's, R_2.
- Characterize the equilibrium level of rent seeking engaged in by two firms who could realize a profit of Π dollars if they win the monopoly privilege.

iii. Discuss why behavior in such games tends to be suboptimal from the point of view of the participants and society at large. Are their cases where rent-seeking efforts do not generate a dead weight loss?

X. Mixed Strategy Equilibria: "Random" Strategies

A. In mixed strategy equilibria, **unpredictable play allows an equilibrium** to occur where certain play does not..

- As we will see in the next lecture or so, many "finite" games that **do not** have a Nash Equilibrium in Pure Strategies.
- However, if random play is allowed, there is always an equilibrium in mixed strategies.

B. A **mixed strategy** is a probability function defined over a player's strategy set.

- The players select among probability functions.
- (Note that a pure strategy can be regarded as a probability function where the probability of a particular strategy is 1, and the probability of all the other strategies is zero.)
- One of the most familiar games with a mixed strategy equilibrium is the "paper, rock scissors" game that many of us played when we were children.

C. To calculate payoffs when using mixed strategies, players are normally assumed to maximize their **expected utility** (the expected payoff) associated with their mixed strategy.

- For example, if there are four possible payoffs, U_1, U_2, U_3, and U_4 and the probability of realizing U_1 is P_1, probability of realizing U_2 is P_2, the probability of realizing U_3 is P_3 and or realizing U_4 is P_4.
- The expected payoff is $U^E = P_1U_1 + P_2U_2+P_3U_3+P_4U_4$
- For a more complete discussion of expected Utility see section XI below.

D. Consider the equilibrium of what Buchanan calls the *Samaritan's dilemma* (see Buchanan, 1972, Tullock, 1983, or Rasmussen 1994, p. 68).

Samaritan's Dilemma	Bum Work	Loaf
Donor Aid	3 , 2	-1, 3
No aid	-1, 1	0, 0

i. There is no pure strategy equilibrium to this game.

- *Consequently, it would not be rational for either player to allow their behavior to be entirely predictable.*

ii. To explore the possibility of a mixed strategy equilibrium, suppose that the probability of the Bum working is w and that the probability that the donor will aid the Bum is g.

iii. The Samaritan's expected payoff from participating in this game is:

$$U_S^E = g\,[\,3w + (1\text{-}w)(-1)\,] + (1\text{-}g)\,[\,(-1)w + (1\text{-}w)0\,]$$

- (Recall from statistics that the **joint probability** that **g** and **w** both happen is **gw** as long as g and w are chosen independently of one another.)

iv. Differentiating with respect to g and solving allows us to characterize g*

- $[\,3w + (1\text{-}w)(-1)\,] - [\,(-1)w + (1\text{-}w)0\,] = 0$

- Note that g is not included in this equation!

v. This suggests that g does not have a simple "interior solution."

- Solving for w, using the first order condition => [3w - 1 +w] + [w] = 0 => 5w = 1
- Thus, we find that $dU^E/dg = 0$ for all g if w* = 0.2
- Notice that if *w > 0.2*, then utility always increases with g, so g* should be set at its highest value, 1
- And, if *w < 0.2* then utility always falls with g, so g" should be set at its lowest value, 0.

vi. A similar calculation for the Bum yields

$$U_B^E = w [2g + (1-g)(1)] + (1-w) [3g + 0 (1- g)]$$
or
$$U_B^E = 2gw + w - gw + 3g - 3gw = -2gw + w + 3g$$

vii. Differentiating with respect to g, we find that the first order condition for w*:

- -2g + 1 = 0 which implies that $U_B^E = 0$ only if g = 0.5
- e. g. if the Samaritan sets g = .5 , then any value of *w* will generate the same utility for the Bum.
- However, if g> 0.5 the Bum's utility falls as w increases, so w* should be set at its lowest value, in which case w*= 0.
- If the Samaritan sets g< 0.5 , the utility rises as w increases, and w should be set at its highest value, in which case w*=1.

viii. [Draw the implied "very strange looking" reaction functions.]

E. Note that if the Donor gives with probability 0.5 and the Bum works with probability 0.2, both are in equilibrium. Both are on their "best reply function" and neither has a reason to change their mixed strategy.

i. Neither player can do better than this combination. (See your diagram of the reaction functions.)

ii. No player can achieve a higher expected payoff by changing his or her strategy.

iii. (Moreover, using any other strategy is likely to be worse, *if that alternative strategy is discovered* by the opponent.)

F. One peculiar property of most **mixed strategy equilibria** is that at the equilibrium each player is *indifferent* among other (non-equilibiurm) probabilities given the other's behavior, but *no other probability* combination generate an equilibrium.

i. *Given* the behavior of other player(s), each player is indifferent among all strategies because they all generate the same expected payoff (expected utility)

- (Recall the first order conditions above.)

ii. This equilibrium is also a bit peculiar in that it assumes that all players understand the entire game, not simply their part of it.

- This requires them to "know" the other player's strategies, payoffs, and method for making decisions in a setting of uncertainty.
- At a Nash Equilibria in pure strategies, players do not have to understand the behavior of other players to realize that they should not change their strategies.

iii. In other words, achieving a "Mixed Equilibrium Strategy" requires a very sophisticated **consciously game theoretic mode of thinking** by the players (a bit more like a Stackelberg model).

- All players have to understand the nature of the game and anticipate the rational responses of other players.
- To do this, they must know a lot about the other players.
- [Note that if one of the players mistakenly plays a pure strategy, no one is made worse off, but if this pure strategy players is discovered the other players will want to change their strategies. Why? (Hint: think of the paper, rock scissors game.)]

iv. Of course, the same mode of reasoning applied to a game with a deterministic equilibrium implies that players will go directly to the Nash equilibrium.

- Fortunately, in many cases the Nash equilibrium *may emerge* from a series of "moves" even if the players behave "myopicly" rather than fully rationally.
- (Myopic players optimize only one move at a time, rather than taking account of all possible future moves of opponents.)
- (See diagram of myopic adjustment in, for example, the lottery game.)

G. Our results demonstrate that **equilibrium mixed strategies can be computed by solving for a probability function that makes the other player(s) indifferent among their strategy choices**.

- [Student Puzzle: fine the mixed strategy equilibria for a "zero sum" game; and for the Paper, Stone, Scissors game played by children. How do your choices of payoffs affect the "mixing probabilities?"]
- [Student puzzle: It can be said that a mixed strategy equilibrium depends on "random play" being "perfectly predictable" How is this possible?]

XI. Additional Notes on Rational Choice in a Stochastic Environment

A. The mixed strategy equilibrium is only one of many settings in which choices must be made in an environment in which particular outcomes are not known at the moment choices have to be made.

- The simplest cases are coin tosses, rolling dice, and lotteries.
- More complex cases arise whenever a long term effect has to be estimated in some way, as with future interest rates, weather, or political contests.

B. Probabilities. One way to characterize a stochastic environment is with a **probability function** mapped over the "event space."

i. A probability function has several important properties.

ii. Every **possible outcome** is assigned a number between 0 and 1.

- If the possible outcomes are $X_1, X_2, X_3 \ldots X_N$ then a **probability function** defined over those possibilities looks like $P_i = f(X_i)$, for i = 1, 2, ... N.
- Probabilities cannot, by definition, be a negative number.
- *Every impossible outcome is assigned the value 0.*

iii. The sum of the probabilities, by definition, has to add up to 1.

- $(\Sigma^N_{i=1} P_i)= 1$

C. The "**expected value**" of a random event such as X above is a technical statistical term that means "average" outcome.

- It is widely used in rational choice models and in applied work by statisticians.
- The expected value of X can be calculated as: $X^e = \Sigma^N_{i=1} P_i X_i$
- For example in rolling a dice, Pi = 1/6 for i = 1, 2 ... 6 and $X_1 = 1, X_2 = 2, \ldots X_6 = 6$

- $X^e = \Sigma^N_{i=1} P_i X_i$ = (1/6)(1) + (1/6)(2) + (1/6)(3) + (1/6)(4) + (1/6)(5) + (1/6) 6 = **3.5**
- (As this illustration illustrates, sometimes the expected value is actually impossible. However, if one calculated the average of a large number of dice throws (say 25 or more) the result would be number quite close to 3.5.)

D. Expected Utility. For many purposes, in rational choice theory only the "ordinal" properties of utility functions matter.

- That is to say, the fact that U(A) = 2 * U(B) only means that A is preferred to B, not that A is twice as desirable as B.

i. However, **to calculate expected utility** in the conventional ways, requires utility functions for which such algebra has meaning.

- Only a subset of possible utility functions can be used to calculate "expected utilities."
- These are called Von Neuman Morgenstern utility functions.

ii. **Von-Neuman Morgenstern** utility functions are bounded and continuous.

- Von-Neuman Morgenstern utility functions are "unique" up to a linear transformation.
- Consequently, they are regarded by many economists and decision theorists to represent *quais-cardinal* utility (in addition to ordinal utility).
- V-N Morgenstern Utility functions have the property that if $U^e(A) > Ue(B)$, then A is preferred to B, where $U^e(A)$ is the "expected utility of stochastic setting A" and $U^e(B)$ is the expected utility of stochastic setting B.

iii. The *expected utility* associated with such a stochastic setting is calculated as the expected value of utility:

$$E(U(X)) = U^e = \sum_{i=1}^{N} P_i \, U(X_i)$$

- where the *value of the possible outcomes is now measured in utility terms.*
- E(U(V)) is more **commonly written as $U^e(X)$**

iv. It is important to remember that $U^e(X)$ is **not** generally equal to $U(X^e)$.

- $U^e(X)$ is **the same as** $U(X^e)$ **only if U is a linear** function of V.
- In such cases, the individual is said to be **risk neutral**.

v. DEF: An individual is said to be **risk averse** if the expected utility of some gamble or risk is less than the utility generated at the expected value (mean) of the variable being evaluated.

- A *risk neutral* individual is a person for whom the expected utility of a gamble (risky situation) and utility of the expected (mean) outcome are the same.
- A risk preferring individual is one for whom the expected utility of a gamble is greater than the utility of the expected (mean) outcome.

vi. The more risk averse a person is, the greater is the amount that he or she is willing to pay to avoid a risk.

E. A Von Neumann Morgenstern Utility function can be constructed as follows.

i. Assume that Xmin and Xmax are the best and worse payoffs of the game.

- Recall, in a finite game, all the payoffs are finite, so Xmin and Xmax exist, and are often very easy to calculate.

ii. The utility of Xi where Xmin<Xi<Xmax can be calculated as a "convex combination" of the arbitrary values U(Xmin) and U(Xmax) where U(Xmin) < U(Xmax).

- Recall that a convex combination of X and Y can be written as Z = (1-a) X + (a) Y where 0<a<1.
- Note that "a" can thus be interpreted as the probability of X, and Z as the expected value of getting X with probability "a" and Y with probability "1-a."

iii. Note also that this "probability" can be varied until the individual of interest is indifferent between the gamble and Xi.

- The utility value of Xi is then defined to be:
- U(Xi) = p U(Xmin) + (1-p) U(Xmax).
- (That is to say, assign utility numbers to events (Xi) by varying P until the person of interest is indifferent between the event (Xi) with certainty and the gamble over Xmin and Xmax defined by P.)

iv. The Von-Neumann Morgenstern utility index was worked out in their very important game theory text *The Theory of Games and Economic Behavior*, 1944.

- (Student puzzle: Why does the difference between ordinal and cardinal utility matter?)
- (Student relief: for most of the rest of this course, we require only "ordinal" utility functions.)

XII. Exercises and Problems

A. Create a zero sum game 2x2 game using the payoffs of -1 and 1 in various orders.

i. Does your game have a Nash equilibrium in deterministic strategies?

ii. Does your game have any Pareto optimal outcomes?

iii. If there is a deterministic equilibrium, change your payoffs to eliminate this equilibrium.

iv. Find the mixed strategy equilibrium for your game, and explain why it is an equilibrium.

B. Repeat "A" with a two person three strategy game.

C. Repeat "A" again using zero sum payoffs from -3 to +3.

D. Create a two stage game in which in stage one, players may send signals (creditable commitments) to each other about whether they will cooperate or not in the PD game that follows.

i. Assume that creditable commitments are possible, but that creating them is costly.

ii. Find the equilibrium strategies in stages 1 and 2 of this game.

iii. Can communication in this type of setting solve the PD game?

iv. How costly does a creditable commitment have to be to make a commitment to cooperate a poor strategy?

v. How might a game be designed to make "truth telling" an optimal strategy? (Discuss a couple of examples from ordinary markets regarding quality or price.)