I. An Introduction to the Mathematics and Rational Choice.

Scope of Course, Usefulness and limitations of deductive methodology. Basic Concepts: compact sets, convexity, continuity, functions, Application: preference orderings, fundamentals of consumer theory.

A. Essentially all scientific work attempts to determine what is general about the world.

i. For example, successive sun rises may be more or less beautiful but all sun rises on Earth are caused by the Earth's daily rotation in combination with light generated by our nearest star..

ii. Persons take account of many different characteristics of an automobile when they select a car, but all at some point all must consider the price of the car. Economics argues that the higher is the price the less likely a given person is to purchase a particular car, other things being equal.

B. This process of finding relationships or general rules of thumb is complex but may itself be described as a joint exercise in logic (model building) and observation (empirical testing)--or perhaps more accurately as a rough cycle of model building, empirical testing, refinement, retesting, and so, forth.

i. This course attempts to provide students with the core mathematical concepts and tools that are the most widely used by economists in the "model building" part of the scientific enterprise.

ii. Econometrics will introduce you to the core statistical methods used in the "testing" part of economic science. For the most part, these tools are mostly involve the concepts and methods of mathematical optimization, and various methods for applying them to settings where rational decisions are, or can be, made.

C. Not all models are mathematical, but mathematical models have many advantages over other modeling methods. Two of the most important are LOGICAL CONSISTENCY (what you have derived is true, given your assumptions) and CLARITY (you know, or should know, what you have assumed).

i. When the rules of logic are applied to numbers the result is mathematics.

ii. Most of the mathematics we have been taught can be deduced from a few fundamental assumptions using the laws of logic. (See the postulates of Peano, an Italian mathematician (1850 - 1932).)

D. An additional advantage of mathematical models is that they allow mathematics (deduction) to be used as an "engine of analysis."

i. If the model is "close enough" then properties deduced from the models will also be properties of the real world.

ii. That is to say, if model "Y" implies that "X" occurs whenever "Z" is present, then if Y is an exactly representation of the real world, and we see that "Z" is present, then we should obvserve "X" in the real world as well as in the model.

iii. If model "Y" is a "more or less" realistic representation of the real world, a rough approximation, then if we see "Z" then we should be more likely to see "X" than "~X" if Y is a useful model.

E. Economists often attempt to deduce some consequences of purposeful decision making using "lean models," models that make very few assumptions about decision making in a setting of scarcity.

i. Economists usually assume that fundamental properties of individual purposes or goals can be represented with preference orderings or utility funtions.

ii. In addition, a choice setting is generally characterized with a feasable set of some kind--as with a budget set or budget line, strategy set, or technological production possibility set or function. , or conditional probability fun

iii. In some cases, opportunities and technologies may be assumed to have specific random probability distributions if the world is believed to be stochastic in the area of interest or if individual have imperfect knowledge about how the world operates in the area of interest..

iv. As will be seen in this course, the design of economic model is partly based on economic considerations, partly on mathematical ones, and partly on "professional conventions.".

II. Some Fundamental Concepts and Definitions from Mathematics

A. Some fundamental properties of logical and mathematical **relationships**:

i. DEF: Relationship R is **reflexive** in set X, if and only if aRa whenever a is an element of X.

ii. DEF: Relationship R is **symmetric** in set X if and only if aRb then bRa whenever a and b are elements of set X.

iii. DEF: Relationship R is **transitive** in set X if and only if aRb and bRc then aRc when a, b, and c are elements of set X.

- Recall that within the set of real numbers, there are several relationships which are symmetric (equality), reflexive (equality) and transitive (equality, greater than, less than, greater than or equal than, less than or equal than).

iv. In economics there are also several relationships which possess all three properties, and some that exhibit only transitivity.

- In general, economists assume that preference orderings satisfy all three of these properties.
- Strong and weak preference orderings are transitive, while indifference is transitive, symmetric and reflexive.
- Indeed, *rationality in microeconomics is often defined as transitive preferences.*

B. DEF: A **function** from set X to (or into) set Y is a rule which assigns to each x in X a unique element, f(x), in Y. Set X is called the domain of function f and set Y its range.

C. DEF: A **utility function** is a function from set X into the real numbers such that iff aPb then U(a) > U(b) and if aIc then U(a)=U(c) for elements of the set X.

i. Note that the indifference relationship, I, can be defined in terms of the weak preference relationship R.

- The weak preference relationship R means "at least as good as."
- Note that if aRc and cRa, then aIc.

ii. Similarly, the strong preference relationship, P, can be defined in terms of the weak preference relationship.

- The strong preference relationship means "better than."
- Note that if aRb but b~Ra then aPb.

D. Note that the assumption that a utility function exists, is equivalent to the assumption that individual preferences are such that *preferences are transitive*;

i. Utility functions also assume that preferences are complete: each bundle (combination of goods and/or "bads") has a unique rank (utility number).

- Every bundle of goods generates either more or less or the same utility level as other goods.

ii. (Some theorists make a distinction between complete and incomplete utility mappings from X to R, but this distinction is not important for "routine" decisions.. Why?)

E. Some important definitions and concepts from Set Theory.

i. DEF: An infinite series, x1, x2, ... xn is said to have a **limit** at x* whenever for any d >0, the interval x* - d, x* + d contains an infinite number of points from the series. (That is to say, x* is a limit point of a series in any case where there are an infinite number of elements of the series arbitrarily close to x*.)

ii. DEF: A set is **closed** if it contains all of its limit points.

iii. Def: A set is **bounded** if every point in A is less than some finite distance, D, from other elements of A.

iv. Def: A set is **compact** if it is closed and bounded.

v. Def: A set is **convex** if for any elements X_1 and X_2 contained in the set, the point described as $(1-\alpha)X_1 + \alpha X_2$ is also a member of the set, where $0<\alpha<1$.

- Essentially a convex set includes all the points directly between points in the set.
- That is to say, any convex (linear) combination of two points from the set will also be a point in the set.
- Thus a solid circle, sphere, or square shaped set is a convex set but not a V-shaped or U-shaped set. What other common geometric forms are convex?
- Example from economics: usually "better sets" are assumed to be convex sets. That is to say, the set of all bundles which are deemed better than bundle a is generally assumed to be a convex set.
- Another example, is the budget set, the set of all affordable commodities give a fixed wealth and fixed prices for all goods that might be purchased.

F. Convexity and compactness assumptions are widely used in models of human decisionmaking. For example, opportunity sets and production possibility sets are nearly always assumed to be convex and compact.

G. Some important concepts and definitions from **Calculus**

i. Def: Function Y = f(X) is said to be **continuous** whenever the limit of f(X) approaches Y= f(Z) as X approaches Z.

- Or alternatively, function Y = f(X) is said to be continuous if for every point in the domain of X, and for any e >0, there exists d > 0, such that |f(X) - f(Z)| < e for all X satisfying |Z - X| < d.
- (That is to say, points only a finite distance from Z should generate function values within a finite distance of f(Z). In fact, f is continuous if for any finite distance e (epsilon) there exists d (delta) such that any value within delta of z generates a function value within epsilon of f(z).)

ii. Def: the limit of a function: function f is said to have a **limit point** y* at x* if and only if (iff) for every e > 0, there is a d > 0 such that |f(x) - y*| < e whenever |x - x*|<d.

- The limit of f at x* is denoted $\lim_{x \Rightarrow x^*} f(x) = y^*$
- If there is a real number y* satisfying this definition at x*, we say that the limit of f at x* exists.
- Note that this definition rules out the existence of different right hand and left hand limits. (why?)

iii. Def: function f is said to be **differentiable** if and only if (iff) for every x contained in set X the limit point of { [(f(x) - f(z)]/(x - z) } exists.

- Note that if f is differentiable, f is also continuous. (why?)

H. Within microeconomics, utility functions and production functions are generally assumed to be continuous and twice differentiable.

i. Such assumptions clearly rule out some kinds of *decision makers* just as the assumption that production possibility sets and opportunity sets are convex and compact rule out some kinds of *choice settings*.

ii. These assumptions are made largely for "economic" rather than "empirical" reasons. That is to say, generally it is felt that the benefits of more tractable models overwhelms the costs of reduced realism and narrower applicability.

iii. Of course if continuous versions of the choice settings lead to empirically false predictions, then continuity assumption would be dropped.

iv. When discrete aspects of the choice problem are important, various tools from set theory, integer programming, and real analysis can be applied instead of calculus.

- Its just that in most of the cases of interest to economists, the assumption of continuity is approximately correct. There may be a smallest grain of sand, but it is pretty small!
- Try to think of cases where the simplifying assumptions of continuity and convexity will generate predictions about behavior that are clearly wrong.

v. In game theory, it is often convenient to model a continuous choice environment as if is discrete. This allows some elementary but powerful tools to be used.

- Again, it could be said to be economics as much as empirical reality that dictates the mathematical tools use.
- A model should be reasonbly simple (lean) and reasonably easy to manipulate.

III. Problem Set

A. Suppose that Al always prefers larger apartments to smaller ones, but is unable to discern difference of ten sq. ft. or less. Are Al's preferences transitive? Explain.

B. Determine whether the following sets are convex sets or not.

i. Al has a budget set W > PaA + PbB, where A and B are both non negative numbers. Pa is the price of good A and Pb is the price of good B. Is Al's budget set convex?

ii. Barbara has a bliss point "B" characterized in terms of all goods relevant over which her preferences are defined. Consider bundle "C" which has less of every good than B. Is the better set for "C" convex? (Construct two two-dimensional examples.)

C. Consider the function $f(X) = 1/X^2$ for $X \neq 0$ and $f(X) = 1$ for $X = 0$.

i. Is the domain of f compact?

ii. Is f a continuous function?

iii. Is f monotone increasing for x>0 ?

iv. Prove that the f does not have a limit at X=0.

D. For most purposes, economists assume that utility functions are continuous and twice differentiable.

i. What does this differentiability imply about the commodity space over which the utility function is defined?

ii. What does twice differentiability imply about the shape of the utility function?

iii. Are their any important limitations of models which rely upon the assumption of differentiable utility functions? Discuss.

iv. Is differentiability an important modeling assumption or a mathematical convenience? Explain.

IV. An Introduction to Optimization

A. The core of modern economics is the notion that individuals optimize. That is to say, individuals use the resources available to them to advance their own personal objectives as well as they know how.

- In consumer theory, the consumer's decision is represented as an effort to maximize utility given a budget set.
- In the theory of the firm, firms are assumed to maximize profit, given production technology and market opportunities.
- In public choice theory, candidates are assumed to maximize votes given the positions of other candidates and the preferences of voters.

B. Such choices can all be characterized with the mathematics of constrained optimization in settings where the individual controls something that may be more or less continuously varied: time, money, policy position, output.

- In each case there is an objective (utility, profit, votes ...) and each case there are constraints which characterize a feasible set (budget set/line, production function, distribution of voter preferences ...).

V. More Fundamental Concepts and Definitions from Mathematics

A. Many of the mathematical properties of a choice can be deduced from the "shape" of the functions used to model it.

i. One of the most widely used notions of shape is *concavity*. Here are three notions of **concavity**:

ii. DEF: Strictly Concave: function f is *strictly* concave iff

$$\alpha f(X_1) + (1-\alpha)f(X_2) < f(\alpha X_1 + (1-\alpha)X_2) \quad \text{where } 0 < \alpha < 1.$$

(That is, iff all the points on a cord connecting two points on a function lie beneath the function.)

iii. DEF: Concavity: function f is concave iff

$\alpha f(X_1) + (1-\alpha)f(X_2) \leq f(\alpha X_1 + (1-\alpha)X_2)$ where $0 \leq \alpha \leq 1$.

(Points on a cord connecting two points of a function lie beneath or on the function.)

iv. DEF: Quasi-Concave: Concavity: function f is *quasi* concave

iff $f(X_1) \leq f(\alpha X_1 + (1-\alpha)X_2)$ where: $0 \leq \alpha \leq 1$

and $f(X_1) < f(X_2)$.

(The function lies above, or not below, the lower of the two end points of a cord connecting points on the function.)

v. Any strictly concave function is also a concave function, but not vice versa.

vi. Any concave function is also a quasi concave function, but not vice versa.

B. Economic analysis generally models both individuals and firms as maximizers.

- Individuals maximize utility.
- Firms maximize economic profit.
- As it turns out, in most cases there are theoretical and/or empirical reasons to believe that both utility functions and profit functions are concave or strictly concave.

C. Concave functions have a number of useful properties in the context of "maximizing" behavior.

i. A *strictly* concave function has at most one global maximum.

ii. A **concave function** may have an infinite number of global maxima, but if there is more than one maximum, they make up a continuous interval. (A horizontal line is concave.)

iii. DEF: The **global maximum** of a function, f(x), has a value which exceeds all others over the entire range of the function (e. g. for every neighborhood of x*).

iv. DEF: The **local maximum** of a function, f(x), has a value which exceeds those of other points within a finite neighborhood of x*. That is, f(x*) is a local maximum if $f(x^*+e) < f(x^*)$ and $f(x^*-e) < f(x^*)$ for $0<e<E$, for some $E>0$.

D. Derivatives of functions can be used to characterize *sufficient* conditions for global maxima and minima, as well as concavity and strict concavity.

i. A differentiable function is at a **local or global maximum or minimum** whenever its first derivative has the value zero.

ii. Given that one is at an extremal (an f(x) where the first derivative, f'(x), is zero),

- the extremal is a local maximum if the second derivative is negative,
- the extremal is a local minimum if the second derivative is positive.

iii. A function is *strictly concave* if its second derivative is negative over its entire domain.

iv. A *strictly* concave function has at most one global maximum. Thus, if function f is strictly concave and f'(x*) = 0, than f(x*) is the global maximum of f.)

v. A function is *concave* if its second derivative is less than or equal to zero over its entire domain.

vi. A matrix of partial derivatives, called a *Hessian*, can be used to determine whether a multidimensional function is concave. At a local maximum, the Hessian will be negative definite. (See the Matrix Magic Handout.)

E. Some Useful Derivative Formulas

i. $y = ax + C$		$dy/dx = a$
ii. $y = ax^b + C$		$dy/dx = bax^{b-1}$
iii. $q = ax^by^c$		$dq/dx = abx^{b-1}y^c$
iv. $u = \log(x)$		$du/dq = 1/x$
v. $h = y(x)\,z(x)$	product rule	$dh/dx = (dy/dx)(z) + (y)(dz/dx)$
vi. $h = y(z(x))$	composite function rule	$dh/dx = (dy/dz(x))(dz/dx)$

F. To increase the generality of their models, economists often use very abstact functions that are characterized only by partial derivatives and/or monotonicity and concavity.

i. Generally economists *assume* that many of the functions of interest (utility functions, profit functions, production functions) are twice differentiable, concave, and monotone increasing over the range of interest.

ii. (The stronger assumption of **strict** concavity is more or less equivalent to assuming that the functions of interest exhibit diminishing marginal returns.)

G. Occasionally, for the purposes of illustration, or in order to mathematically derive a specific functional form for an economic relationship of interest (a demand curve, cost function, etc.), further structure is assumed.

i. For example, a utility function might be assumed to have an exponential or *Cobb-Douglas* form, $U = ax^by^c$, where: x and y are quantities of two goods.

- In a Cobb-Douglas function the exponents sum to one, $b + c = 1$.

ii. At other times somewhat less "concrete" assumptions about functional form are used, but ones that are more restrictive than the assumption of strict concavity.

- For example a production function might be assumed to be **homogeneous** of degree 1.
- **DEF.** A function is said to be *homogeneous of degree k*, if and only if whenever $Y = f(X)$, then $f(\lambda X) = \lambda^k Y$
- A production function that is homogenous of degree 1 exhibits constant returns to scale. Doubling all inputs doubles outputs.
- Cobb-Douglas functions and linear functions through the origin ($Y = ax$) are all homogeneous of degree 1.

iii. Occasionally, utility and production functions are assumed to be *homothetic*, a somewhat more general family of functions than homogeneous functions.

- DEF. A **homothetic function** is a composite function of the form $H = h(Q(a,b))$ where Q is a homogeneous function and $dH/dQ \neq 0$.
- All homogenous functions are homothetic functions but not all homothetic functions are homogeneous.
- Homothetic utility functions have linear income expansion paths. Similarly, homothetic production functions have linear output expansion paths. (*The slopes of the isoquants are the same along any straight line through the origin.*)

VI. Constrained Optimization

A. Many optimization problems facing individuals and firms involve constraints of one kind or another.

i. In such cases, the above "unconstrained" maximizing (or minimizing) technique *can not* be directly applied -- by either the individuals themselves or those attempting to model their behavior.

ii. For example, utility maximizing individuals are constrainted by a "binding" budget constraint.

B. The Substitution Method. In some cases, it is possible to "substitute" the constraint into the objective function (the function being maximized) to create a new composite function which fully incorporates the effect of the constraint.

i. For example consider the separable utility function: $U = x^{.5} + y^{.5}$ to be maximized subject to the budget constraint $100 = 10x + 5y$. (Good x costs 10 \$/unit and good y costs 5 \$/unit. The consumer has 100 dollars to spend.)

ii. Notice that, from the constraint we can write y as: $y = [100 - 10x]/5 = 20 - 2x$

iii. Substituting into the objective function (in this case, a utility function) yields a new function entirely in terms of x: $U = x^{.5} + (20 - 2x)^{.5}$

- This function accounts for the fact that every time one purchases a unit of x one has to reduce his consumption of y.

iv. Differentiating with respect to x allows the utility maximizing quantity of x to be characterized as:

$$d[x^{.5} + (20 - 2x)^{.5}]/dx = .5x^{-.5} + .5(20-2x)^{-.5}(-2)$$

- This derivative will have the value zero at the constrained utility maximum.
- Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for x yields:

$$4x = 20 - 2x \Rightarrow 6x = 20 \Rightarrow x^* = 3.33$$

- Substituting x* back into the budget constraint yields a value for y*

$$y = 20 - 2(3.33) \Rightarrow y^* = 13.33$$

v. No other point on the budget constraint can generate higher utility than that at $(x^*,y^*) = (3.33,13.33)$.

VII. Problems

A. Consider the demand function $Q = a + bP + cY$, with $b < 0$ and $c > 0$.

i. Find the slope of this demand function in the QxP plane.

ii. Find the slope of this demand function in the PxY plane.

iii. Show that this demand function is homogeneous in prices iff: $a = -cY$.

iv. Is the associated revenue function (R=PQ) concave? strictly concave? (Hint: use the inverse demand function to characterize the price at which the firm can sell its output.)

v. What is the revenue maximizing quantity of this good?

vi. Prove that this demand function is continuous in Y.

B. Use the substitution method to:

i. find the utility maximizing level of goods g and h in the case where

$$U = g^a h^b \text{ and } 20 = g + h,$$

ii. find the utility maximizing bundle of goods when $40 = g + h$ (i.e. if the wealth constraint is twice as high).

iii. characterize the profit maximizing output of a firm where $\Pi = pQ - C$ and $c=c(Q, w)$.