An Introduction to Non-Cooperative Game Theory

Applications of game theory are commonplace in the most active areas of research in economics, political science, and philosophy. A quick look at any economics, political science, and sociological journal in the past two decades will reveal a large number of articles that use game theory to analyze a broad range of individual behavior in wide a variety of social settings.

The use of game theory in economics is quite old. For example, the Cournot duopoly model (1838) is an example of a non-cooperative game with a Nash equilibrium. Analysis of Stackelberg duopoly (1934) and monopolistic competition (Chamberlin 1933) are also based on models and intuitions like those of game theory, although "game theory" did not really emerge as a field of study until after WWII.

In economics, interest in modern game theory increased rapidly during the 1980s and 1990s, and became one of the main methods of analysis in micro economics and experimental economics.

Modern work on: the properties of contracts, team production, negotiations within cartels, credible commitments, the production of public goods, externalities, time inconsistency, bargaining, and models of political and social competition all use game theoretic models as their "engines of analysis."

Most of these applications apply the rational choice model that we have been exploring during this course. That is to say, game theory assume that players are "rational" in the sense that they are forward looking and attempt to maximize net benefits (or utility). The game "outcomes" are normally "payoffs" that are characterized with net benefits or utility levels. Each player is assumed to maximize their own utility or net benefits, given what the other players have or might do.

In this next block of the course, we'll review some basic tools from game theory and apply them to micro economic phenomena, mostly ones dealing with market structure and team production.

As in the rest of the course, we'll use the rational choice model to "predict" behavior, but now in settings where one's payoff depends in part on the behavior of other persons "in the game."

As usual, we will begin with relatively simple choice settings and proceed to more complicated ones.

Initially, we'll assume that there are just two players and two possible strategies. This turns out to be enough to shed light on a variety of choice settings. If we have time, we'll also take a quick look at more complex settings with more possible strategies, larger numbers of players, and repeated games.

I. Non-Cooperative Games

A. Game theory can be used to model a wide variety of human behavior in small number and large number economic, political, and social settings.

B. The choice settings in which economists most frequently apply game theory, however, are small number settings in which outcomes are jointly determined by the decisions of a handful of independent decision makers.

i. In "non-cooperative game theory" individuals are normally assumed to maximize their own utility **without caring about the effects of their choices on other persons in the game**.

- The outcomes of the game, however, are jointly determined by the strategies chosen by all players in the game.
- *Consequently, each person's welfare depends, in part, on the decisions of other individuals "in the game."*

C. The simplest game that allows one to model social interdependence is a two-person game each of whom can independently choose between two strategies, S_1 and S_2.

i. There are four possible outcomes to the game:

- (1) both players may choose S_1, (2) both may choose S_2
- (3) player A may choose S1 and player B may choose S2, or (4) vice versa.

ii. The particular combination of strategies is the result of the independent decisions of the two players, A and B (Al and Bob).

D. For example, consider the "**trading game**" below.

i. Bob has bananas and Al has apples. Bob is thinking trading some bananas for some of Al's apples. Al is thinking about trading apples for some of Bob's bananas.

ii. Since trade is voluntary, nothing happens unless both players agree to trade. However, for the purpose of illustration, It is assumed that it costs "one util" or one dollar to make an offer, whether taken or not.

iii. Thus the lower left hand and upper right-hand cells have payoffs for Al and Bob that are lower for the one making the offer (trading), while the other is unaffected.

iv. Normally the payoffs are in terms of "utility," "euros," or "dollars," but occasionally other values are natural for the problem at hand.

v. (This type of game is sometimes called an "assurance game" or a Stag-Hunt game.)

		Bob	
		Trade	**Don't**
Al	**Trade**	**(a, b)** **(10,10)**	**(a, b)** **(1, 2)**
	Don't	**(2,1)**	**(2,2)**

E. A game can be said to have a **Nash Equilibrium** when a strategy combination is "stable" in the sense that no player can change his strategy and increase his or her own payoff by doing so.

i. Note that the above trading game **has two equilibria**, (trade, trade) and (don't, don't). Neither person can make themselves better off by changing their strategy (alone) given that of the other player(s) in the game.

Someone has to "make an offer" to induce trade to take place and making an offer is cosly, making one a bit worse off if it is refused.

ii. Learning to trade takes some time and is sometimes the process is formalized by routines or stable strategies..

Suppliers often take this first step by putting their "wares" out for sale at markets or in shops or on the Web.

The idea of a "store front" can be thought of as an institution for making offers to potential customers.

This technology for solving the assurance game is very old, and goes back beyond Rome and beyond Ancient Greece.

iii. **DEF:** A state of the world or game outcome is said to be **Pareto Optimal** or Pareto Efficient, if it is impossible to reach another state where at least one person is better off and no one is worse off.

Note that the (Trade, Trade), equilibrium is Pareto optimal, but none of the other outcomes are.

iv. A shift from one cell to another is said to be a **Pareto Superior Move**, if at least one person is better off and no one is worse off.

A move from the (don't, don't) cell to the (trade, trade) cell is a Pareto superior move.

F. Two-Person two-strategy games are often used to illustrate decisions that may in reality involve more than two persons and involve more than two strategies.

i. Some choices can be "factored" down into a series of two person choices of this sort.

ii. Other cases, involve choice settings where the 2x2 game captures the essence of the choices involved.

iii. 2x2 and 2x3 games can often capture the essential features of important choice settings of interest to social scientists.

a. However, as the above game theoretic representation of the "problem of exchange" demonstrates, the usual economic representation of exchange misses some details that may be important.

b. On the other hand, the Edgeworth box very nicely illustrates why the trade, trade equilibrium tends to be Pareto optimal, which we used to determine the relative sizes of the game's payoffs.

II. The Prisoners' Dilemma Game: A Simple Non-Cooperative game, with "suboptimal" outcomes.

A. The *Prisoners' Dilemma game* is probably the **most widely used game** in social science.

i. The "original" prisoners dilemma game goes something like the following. Two individuals are arrested under suspicion of a serious crime (murder or theft). Each is known to be guilty of a minor crime (say jay walking), but it is not possible to convict either of the serious crime unless one or both of them confesses.

ii. The prisoners are separated. Each is told that if he testifies about the other's guilt that he will receive a reduced sentence for the crime that he is known to be guilty of.

iii. The Nash equilibrium of this game is that BOTH TESTIFY (or Both CONFESS).

B. To see this consider the following game matrix representing the payoffs to each of the prisoners:

i. Each cell of the game matrix contains payoffs, for A and B, in years in jail (a bad). [

- Most games have net benefits rather than losses as payoffs, and PD games can be represented in terms of net benefits as well.
- Usually higher numbers are to be sought out, but in this case, higher numbers are to be avoided. They are losses rather than net benefits or utility.)

Classic Prisoners Dilemma (PD) Game			
		Prisoner B	
		Testify Against A	**Don't**
Prisoner A	**Testify against B**	**(10,10)**	**(1, 12)**
	Don't	**(12,1)**	**(2,2)**

ii. Each individual will rationally attempt to minimize his jail sentence.

- Note that regardless of what Prisoner B does, Prisoner A is better off testifying. 10 < 12 and 1 <2. Testifying is the **dominant strategy**.
- Note that the same strategy yields the lowest sentence for Prisoner B. If A testifies, then by also testifying B can reduce his sentence from 12 to 10 years. If A does not testify, than B can reduce his sentence from 2 to 1 year by testifying. The dominant strategy is a **pure strategy** in that only one of the strategy options is ever "best" given the options of the other player, and that "pure strategy" is used by each player.

iii. The (testify, testify) strategy pair yields 10 years in jail for each.

- This is said to be the Nash equilibrium to this game, because given that the other player has testified, each individual regards his own choice (testifying) as optimal.
- No player has an incentive to independently change his own strategy at a Nash equilibrium.

iv. That is to say, every player is doing as well as he or she can (e.g. maximizing net benefits or utility) with their chosen strategy, given what all other players are doing.

v. It is a dilemma because each prisoner would have been better off if neither had testified. (2 < 10).

- A Pareto superior move exists. Independent rational choices do not always achieve Pareto optimal results.
- [f course, society at large may regard this particular dilemma as optimal insofar as two dangerous criminals are punished for real crimes.
- (What I have called "testifying" is often called "confessing" in other textbook discussions of PD games.)

C. The prisoner's dilemma game (PD game) can be used to model a wide range of social dilemmas. One game of particular interest for the purposes of this course is to model the behavior of markets with small numbers of firms, such as a duopoly, with just two firms.

i. Note that Bertrand duopolists may behave as if they were perfectly competitive firms at their price setting Nash equilibrium.

ii. This is not the only possible way to represent relations between two firms, but it is a useful one for understanding why cartels are hard to establish.

Bertrand Pricing Game Among Duopolists (the Cartel Enforcement Problem)			
		Bopex	
		Pb = MC	**Pb = Cartel Pricee**
Acme	**Set Price A equal to MC**	**(a, b)** **(0,0)**	**(a, b)** **(6, -3)**
	A Sells at Cartel Price	**(-3, 6)**	**(4,4)**
Assumes a rising MC curve over the range of interest.			

iii. (The **Cournot Duopoly** model where quantity is controlled rather than price was developed in class has a quite different equilibrium--one that is between the monopoly and competitive outputs, prices, and profits.)

Cournot Output Decision				
		Firm B		
		Q_1	**Q_2**	**Q_3**
Firm A	**Monop Q_1**	**10 , 10**	**4, 12**	**1, 11**
	Q_2	**12, 4**	**5, 5**	**2, 4**
	Comp Q_3	**11, 1**	**4, 2**	**0, 0**

iv. There are many PD-like games (social dilemmas) that are of interest to economists--some of which are illustrated in the notes below and/or in the appendices. Examples include:

- The Cournot Duopoly Game
- The team production problem
- Decisions to engage in externality generating activities. (Pollution)
- Competition among students for high grades vs. leisure in universities
- Contract Breach/Fraud (in a setting without penalties)
- Commons Problems
- Public goods problems
- The Not In My Backyard Problem
- The free rider problem of collective action.
- The dilemma of thieves
- The international regulation dilemma
- The arms race

v. **[We will go over some of these games in lecture.]**

- Several other named games are illustrated in the appendices of this set of lecture notes.
- Some of these will turn up in your other courses, where they will be a "main dish" rather than a "side dish."

D. What characterizes a PD game is that the "cooperate, cooperate" solution is preferred by each player to the "defect, defect" equilibrium. However, the value generated by defecting is somewhat higher than the cooperative strategy regardless of whether the other player cooperates or not.

i. **Often the payoffs are represented "ordinally" with numbers indicating rank order or utility levels.**

ii. **The higher numbers indicate higher net rewards, rather than with years in jail, or a particular payoff value in profits or net benefits.**

- (3,3) is often used for the mutual cooperative solution and (2, 2) for the mutual defection result.
- The other payoffs are then (1,4) and (4,1) with the defector receiving 4 and the cooperator 1.
- The best outcome is 4, second best is 3, third best is 2, and worst is 1.
- [These payoffs are often used in some of the illustrations in lecture.]

E. The PD payoffs can also be represented algebraically with (abstract) payoffs.

- (C, C) and (D, D) are the payoffs of the mutual cooperation and mutual defection outcomes

- And (S, T) and (T, S) for the "temptation" and "sucker's" payoffs when one person defects and the other is "played for a sucker.
- In a PD game, T>C>D>S.

F. The PD game's main limitations as a model of social dilemmas are its assumptions about the number of players (2), the number of strategies (2), the period of play (1 round).

- However, most of these assumptions can be changed without changing the basic conclusion of the analysis.
- It is the generality of the conclusion, rather than generality of the model itself that allows it to be used to illustrate so many choice settings.
- For example, similar conclusions follow for N-person games in which the players have an infinite numbers of strategies (along a continuum) and play for any *finite* number of rounds if the net benefit maximizing efforts of A impose costs on B, as they do in the PD game.
- However, **not all games have equilibria that are social dilemmas**, as with the (trade, trade) equilibrium in the first game matrix above.
- And not all social dilemmas are as bad as those associated with the 2x2 version of the PD game.
- [As an exercise, develop payoffs for a 3x3 (3 strategy) game matrix in which the middle solution is a Nash equilibrium, but that one of the other solutions is Pareto superior to it. Not all dilemmas are extreme (corner solution) ones.]]

III. Organizations and Team Production

A. An important game that takes place within every firm and most other organizations is the **Team Production Game**.

B. The term "organization" has a variety of meanings, although all share the notion that organizations are relatively durable products of human action, although not always entirely of human design. An organization is normally created by formeteurs to realize advantages that can be realized when a group of person's can be induced to interact in a manner that increases their overall productivity.

C. Team production, by definition, generates more output than would be produced by the same persons acting independently of one another. When organization's pay, their teams will produce more output than otherwise similar groups that are "not organized."

D. Organizations must solve a variety of coordination, prisoners dilemma, coordination, and prisoner's dilemma with exit problems. With this in mind, a variety of standing procedures and rules are normally adopted by organizations to reduce such problems. Such rules are often necessary to realize the advantages of team production.

i. In game theoretic terms, an organization's rules create the payoff functions and determines the set of persons eligible to participate in the contest(s) devised or managed.

a. The rules include the procedures for recruiting team members, and a variety of conditional rules that determine compensation and promotion within the organization.

b. As in any "standing game" the organization's rules determine the returns from alternative strategies at the margin, which in turn encourages some forms of behavior and discourages others within the organization.

c. Through such effects, the standing rules determine the equilibrium within the organization, the stable patterns of life that exists within organizations during "office hours."

d. Relatively few of these rules affect behavior while at home, where team members become individuals and family members who may behave entirely differently.

e. (It bears noting, however, that the reach of an organization's rules is being extended as office hours become more elastic, because of internet and cell phone technologies).

E. In many cases, the main reason that organizations are founded is to address is the shirking dilemma, the tendency of persons engaged in team production to shirk rather than work.

i. The team production problem is illustrated in the game matrix below. Note that the payoffs resemble those of a PD game.

ii. Unfortunately, the "natural" reward structure of the "cooperative"--sharing team production equally (and unconditionally)--fails to encourage sufficient productive work to maximize the advantage of team production.

a. Shirking by team members frees time for their own use, while reducing the productivity of other team members.

b. In the game illustrated mutual shirking is the equilibrium outcome.

The Shirking Dilemma of Team Production in Natural Cooperative		
	Team Member B	
	Work	Shirk
Work (A)	3, 3	1,4
Shirk (A)	4,1	2,2

The cell entries are utilities (or net benefits), the rank order of subjective payoffs for the team members (A, B). The dilemma in the "natural case" is that both team members shirk rather than work.

An Illustsrating quote:

"**When working each man for himself**, these men were able to earn higher wages at 3%; cents a ton than they could earn when they were paid 4%; cents a ton on **gang work** ; and this again shows the great gain which results from working according to even the most elementary of scientific principles." *Principles of Scientific Management*, F. W. Taylor (1914: 76) [available at google books]

F. Solving the Team-Production Problem

i. The possibility of creating better incentives for team production creates an incentive for "formeteurs" to create organizations with artificial reward systems. If they do so successfully, they may profit from creating such organizations (firms, sports teams, clubs, etc.)

ii. The game matrix below illustrates one possible solution to the shirking or team production dilemma.

iii. Team production is assumed to be worthwhile, which implies that the productivity of each member is increased by the efforts of the others.

In the game above, which is referred to as the natural cooperative, the group's output is shared equally.

iv. In the game below, a formeteur has created an artificial reward structure for his or her team. Each team member receives a reward (R) for work and a penalty (P) for shirking that is independent of the efforts of other team members.

Organizational Solution to the Shirking/Free Rider Dilemma		
	Team Member B	
Team Member A	Work	Shirk
Work (A)	R, R	R, 4-P
Shirk (A)	4-P, R	2-P, 2-P

v. Team members will avoid shirking if $R > 4 - P$ and $R > 2 - P$. Any combination of rewards and penalties such that R + P > 4 is sufficient to solve this intra-organizational rent-seeking problem.

vi. Any reward greater than 2 is sufficient to attract team members from natural cooperatives. In order to be self sustaining the rewards can be no greater than 3.

vii. **Person who establish organizations (formeteurs) will attempt to solve such team production problems through artifical incentive structures** (rules for rewards and punishments) as long as penalties greater than 2 are feasible. In that case a reward structure with 3>R>2, can potentially yield profits for organizers.

G. If the organization succeeds, the artificial reward structure will increase in team output.

i. In the illustration, the **new rules condition rewards only on the behavior of single team members** and does so in a manner that aligns the interests of individual team members with those of the organization. "Productive" behavior is rewarded and "unproductive" behavior is punished.

ii. If the reward or punishment is high enough, all team members work (advance the organization's interests) rather than shirk. In the illustrating game matrix, an R>2 and P>2 are sufficient to solve the dilemma, and to attract team members from the natural cooperative.

iii. A reward equal to 2.5 for each person and punishment of 1.75 for shirking, produces 6 units of output, and an organizational surplus of 1 unit of output for the formeteurs. (No one shirks in equilibrium.)

H. Organization may be said to **adapt to changing circumstances, it does so when its leaders changing the rules** (usually at the margin) in a manner that changes the equilibrium among its team members and/or that between itself and other organizations.

That is to say, an organization's rules are normally revised through time, as circumstances and goals change.

I. A richer development of team production can be done using a production function with economies of scale to develop the numbers for a 3x3 team production game as done in class.

i. A dilemma is more likely to exist when teams simply share the value they produce and leisure is also valuable. See the class notes for an example.

ii. Such dilemmas can be reduced by paying people for their own specific contribution to the output, although this is not always sufficient for a solution.

iii. As an exercise, create a production function that exhibits increasing returns to scale (rising average product) and assume that leisure has value. Use information from the production function and the benefits from leisure to create payoffs in a game matrix. Find the equilibrium. Is there a dilemma? Why or why not. Repeat with a different production function and/or value for leisure.

Several Game Theory Appendices follow below for interested students.

They will **not be covered on the exam**, **unless** I also cover them in lecture.

They are included for interested students--especially those thinking about graduate school.

IV. APPENDIX (1): A Few Other "Named Games"

A. Several other interesting games can also be created by changing the payoffs of the two player two strategies games..

i. A **zero sum** game is a game in which the **sum of the payoffs in each cell** is always zero. In this game, every advantage realized by a player comes at the expense of other players in the game.

- (Individuals with no training in economics seem to regard all economic activities as zero sum games. Of course, in most cases, exchange creates value for each player. Trade is a *positive sum* game.)

ii. **Coordination games** are games where the "diagonal" cells (top left or bottom right) have the essentially identical payoffs(for example, 1,1) which are greater than those of the off diagonal payoffs, (for example, 0,0).

- Here it is important that some norm be followed by both persons, and either "on diagonal corner" is an equilibrium.
- (All drive on the left side of the road or all on the right have higher payoffs than some drive on each side of the road.)

iii. **Assurance games** are similar to coordination games. The off diagonal payoffs for the "cooperative" strategy are equal to or below those of the on diagonal cells (2, 0), however the upper left-hand "cooperative" cell has a higher value to both players (3, 3) than the lower right-hand "do nothing" cell, the original position (2, 2).

- It will take some kind of guarantee or trust to generate moves from the original lower right hand score to the higher upper left-hand cell.
- The trading game developed above is an assurance game.
- Some game theorists have renamed the assurance game a stag hunt game after a setting described by Rousseau in his *Discourse on Income Inequality* (1755)

> "If it was a matter of hunting a deer, everyone well realized that he must remain faithfully at his post; but if a hare happened to pass within the reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple and, having caught his own prey, he would have cared very little about having caused his companions to lose theirs."

- See Brian Skyrms' book on the *Stag Hunt* for a complete discussion.
- (Note that this translation suggests that the Stag Hunt is really a PD game rather than an assurance game, at least if the starting point is two hunters at

their stag hunting post. In the usual assurance game, the better equilibrium is, of course, stable!)

iv. The games of **chicken** is a game in which coordination is disastrous rather than beneficial.

- As in the assurance game, one of the coordinated out comes is preferred to the other. But in this case, the "off diagonal" strategies yield higher payoffs. (1,1) > (0,0)
- The off diagonal scores are generally higher, although one person does better than the other as with (4,2) and (2,4).
- (The payoffs can be adjusted so that mutual bravery yields an intermediate payoff such as (3,3)).
- (Illustration, the old 1950s teenage drivers game of chicken on rural roads.)

V. APPENDIX (2): Other Applications of Game Matrices to Problems of Interest to Economists and other Social Scientists

A. As noted above, a variety of social dilemma problems can be analyzing Prisoner's Dilemma Games.

i. One such game is the "Public Goods" or free rider problem.

ii. In this game, a public service can be produced by either player alone by paying the full cost of the service, or it can be jointly produced if each pay's for half of the service.

iii. (A **pure public good** is a good that is "perfectly shareable," a good which once produced can be enjoyed by all in the community of interest.)

iv. Suppose that the value of the service is V to both Al and Bob, and the cost of the service is C.

- If both players contribute to the cost of the public good, then each pays C/2.
- If only one does, then that person pays the full cost, C.
- If the public good is produced, then each player receives V.
- This structure of contributions and benefits yields a game with the following "net benefit" payoffs:

A Public Good Game		**Bob**	
		Povide	**Free Ride**
Al	**Provide**	**3,3**	**-1,4**
	Free Ride	**4, -1**	**(0,0)**

v. Note that providing the service makes collective sense but not individual sense, and the good is not produced.

- This is the classic Public Goods problem studied in Public Economics.
- The *Nash Equilibrium of the Public Goods game is mutual free riding*, which generates the (0,0) outcome
- (Note that the payoffs have the same rank order as those in a PD game, but that the motivation for these payoffs comes from the production process assumed.)

vi. This example shows how a common "real world" setting can be represented using a game matrix.

- Of course, in most cases, more than two persons will be involved in paying for or producing the public good, and in most cases the good itself can be produced at various levels.
- Still, the 2x2 representation, captures *essential* features of the "free rider" problem that must be confronted when thinking about the production of public goods.

B. The same matrix can be modified slightly to show **how rewards and penalties can be used to solve such problems**.

i. Suppose that free-riding can be observed, and that penalty P is imposed on any one that free rides.

- The penalty could literally be a fine or a tax imposed on free riders.
- The the penalty might also be non-pecuniary, as with losing the respect of approval of one's friends or neighbors.
- Or, the penalty could be entirely internal, as when a person that violates his or her private rules of conduct anticipates feeling guilty afterwards.

ii. We now incorporate penalty P into the game matrix.

A Public Policy Solution to a Public Good Game		**Bob**	
		Povide	**Free Ride**
Al	**Provide**	**(A, B) (V - C/2, V - C/2)**	**(A, B) (V - C, V -P)**
	Free Ride	**V-P, V - C**	**(-P, -P)**

iii. Given V- C < 0 and 2V > C, there are penalties that will solve the free rider problem.

- For example, any P such that V - C/2 > V - P and P < V - C will do so.

iv. Notice, for example, P = C/2 + 1 is sufficient to solve the problem.

- Smaller penalties may also work such as P = (C+e)/2 with e> 0.
- (Recall that in the problem case, V is greater than C/2.)
- Notice that the penalty has to be higher the larger is the cost of the public good relative to value of the good.
- (There are, of course, a wide variety of penalties that might solve PD games, including ones that involve only "approval" or "shame," but also one that involve the use of police and courts to impose fines or managers to reduce future salaries.)

v. (A similar matrix can be used to illustrate essential features of what economists call Externality Problems.)

C. Two Illustrations of the Regulatory Dilemmas of Neighboring Governments

D. Race to the Bottom. Suppose that are two communities that are interested in regulating some activity within their own territory.

E. Suppose further that regulations in each community affect each other's prosperity, with the community with the "weakest" regulations being somewhat more prosperous than the community with the stronger community.

i. To simply a bit, assume that there are just three types of regulations that can be imposed: weak, medium, and strong regulations.

ii. Suppose also that the joint ideal is "medium, medium"

iii. However, the effect of local regulations (relative to that of the other community implies that each community is a bit better off weakening its regulations, given the other's regulation of the activity of interest.

The Race to the Bottom Dilemma Community B's environmental Regulations			
	weak	**medium**	**strong**
A's env regs weak	**A,B 6,6**	**A,B 8,4**	**A,B 9,2**
medium	**4,8**	**7,7**	**8,5**
strong	**2,9**	**5,8**	**6,6**

iv. Such games have a Nash Equilibrium in Pure strategies that is not Pareto Efficient.

v. This "regulatory dilemma" is sometimes called the "Race to the Bottom" because each government has an incentive to under regulate the phenomena of interest (say air pollution).

vi. Notice also that a voluntary agreement to move to (medium, medium) may not solve the dilemma because it is not a Nash equilibrium.

- It is for this reason that treaties may, for example, have no effect on international air pollution.
- Notice also that this problem can, however, be solved by penalizing weak regulation in some sense.
- This may be difficult to arrange in an international setting although it can be done within a federal system by higher levels of government..

F. NIMBY. Now suppose that the inter-community externality in the opposite direction. That is to say, suppose that the community with the weaker regulation attracts undesirable (say, noisy, ugly, or polluting industries) into the community.

i. Assume again that there are just three levels of regulation and that the two community ideal is (medium, medium) as in the previous example.

ii. In this case, each community is just a bit better off if it has somewhat tougher regulations than its neighbor.

iii. We can just slightly modify the payoffs of the above game to illustrate the new problem.

The Race to the Top Dilemma NIMBY Community B's environmental Regulations			
	weak	**medium**	**strong**
A's env regs weak	**A,B 6,6**	**A,B 4,8**	**A,B 2,9**
medium	**8,4**	**7,7**	**5,8**
strong	**9,2**	**8,5**	**6,6**

iv. This game also has a Nash Equilibrium with dominant strategies that is not Pareto Optimal.

v. This regulatory dilemma is sometimes called the "race to the top" or NIMBY (not in my backyard) problem.

VI. APPENDIX (3): An Illustration of the Essential Mathematics of Nash Equilbria in Games with Continuous Strategy Options

A. There are many settings in which players strategies are not discrete, but rather lie along a continuum of some sort.

- Players on a team may work more or less.
- More or less of a public good may be provided.

B. Such games can be represented mathematically by specifying a payoff (or utility) function that characterizes each player's payoffs as a function of the strategy choices of the players in the game of interest.

C. Consider, for example, a **two-person lottery game** played by two persons. Both want to maximize their "expected" net earnings from purchasing tickets.

i. The **expected value** of an event with outcomes 1, 2, i, ... N is $V^e = \Sigma$ PiVi, where Pi is the probability of event i, and Vi is the value of event i.

- If Al purchases Na lottery tickets and Bob purchase Nb tickets, Al's expected profit is $Ra^e = [Na / (Na + Nb)]Y - Na\ C$ where Y is the prize one and C is the cost of a lottery ticket.
- Similarly Bob's expected net benefit (profit) is $Rb^e = [Nb / (Na + Nb)]Y - Nb\ C$

ii. Al's expected profit maximizing number of lottery tickets can be found by differentiating Ra^e with respect to Na and setting the result equal to zero.

- $dRa^e/dNa = \{[1 / (Na + Nb)] - [Na / (Na + Nb)^2]\}Y - C = 0$ at Na*
- Putting terms over the same denominator and adding C to each side yields:
- $[Na + Nb - Na]/(Na + Nb)^2 = C/Y$ or $Nb/(Na + Nb)^2 = C/Y$
- Next we want to solve for Na
- $Nb = (Na + Nb)^2\ C/Y$ or $Nb(Y/C) = (Na + Nb)^2$
- which implies that $(NbY/C)^{1/2} = Na + Nb$
- so $Na^* = - Nb + (NbY/C)^{1/2}$

iii. This last function is sometimes called a **best reply function**. In this case, it tells Al the expected profit maximizing number of lottery tickets to purchase given any particular purchase by Bob.

- Note that Na* varies with Bob's purchase which implies that Al does **not** have a dominant strategy.

- Note also that a best reply function can be derived for Bob, $Nb^* = - Na + (NaY/C)^{1/2}$

iv. Note also that if both **persons are simultaneously on their best reply function**, neither can change their strategy and improve their payoff (remember that the best reply function for player i maximizes his or her payoff, given the strategies adopted by all other players), as required for the existence of a **Nash equilibrium**.

v. Thus, the **Nash equilibrium** of this lottery game occurs at a point where: $Na^* = - Nb^* + (Nb^*Y/C)^{1/2}$ and $Nb^* = - Na^* + (Na^*Y/C)^{1/2}$

- To find the Na* and Nb* combination where both these conditions hold, one can either substitute the equation describing Nb* in terms of Na into the Al's best reply function and do a bit of algebra.

vi. In a **symmetric game** (a game in which players have the same strategy sets and payoff functions) there is normally a symmetric equilibrium. In this case, the two best reply functions will intersect at a point where Na = Nb.

- At the symmetric lottery game's equilibrium: $Na = - Na + (NaY/C)^{1/2}$
  or $2Na = (NaY/C)^{1/2}$
- Squaring both sides, we have: $4Na^2 = NaY/C$ which implies that $4Na = Y/C$
- or $Na^{**} = Y/4C$ and since Na = Nb at the symmetric Nash equilibrium, we also have $Nb^{**} = Y/4C$

vii. Since each ticket costs C euros, so Al spends Na** C or Y/4 euros on tickets. That is he spend exactly 1/4 of the prize money (if he wins) on tickets.

- [The same is true for Bob, so it is clear that this particular lottery will not be a "money maker" for its organizers.]

D. The lottery game can be generalized to think about a wide variety of games in which one's odds of winning a contest depends upon how much time, energy, wealth, etc. one invests in the game.

E. **Common applications** include the political rent-seeking games, originally developed by Gordon Tullock, legal battles in court, research and development contests by firms, warfare, car racing, grades on university exams, etc..

F. The lottery game and its various applications **can also be generalized** to take account of more than 2 players, and to include "technologies" where the exponents on investments are subject to increasing or decreasing returns.

G. It is surprisingly easy to generalize this game by, for example, including N players rather than two.

i. Let K represent the total investment of the N-1 players, then the expected payoff of a "typical" player is:

- $Ra^e = [Na / (Na + K)]Y - Na\ C$

ii. Differentiating with respect to Na yields:

- $dRa^e/dNa = \{[1 / (Na + K)] - [Na / (Na +K)^2]\}Y - C = 0$

iii. Solving for Na, as above, yields:

- $Na^* = - K + (KY/C)^{1/2}$

iv. This equation is the **best reply function of a typical player** in the present N person game.

v. To find the symmetric equilibrium, note that K = (N-1) Na, so:

- $Na^* = - (N\text{-}1) + [(N\text{-}1)Na\ Y/C]^{1/2}$

vi. solving for Na*, yields:

- $Na^{**} = [(N\text{-}1)/ N^2] (Y/C)$

vii. Note that when N = 2, as above, $Na^{**} = (1/4) (Y/C)$, as before.

viii. The **total expenditure** on "rent seeking" is NC times this amount, or (N-1)Y/N, and this expenditure approaches Y in the limit as N approaches infinity.

H. **Different technologies for increasing one's chance of winning** can also be taken into account by assuming changing our assumptions about investments in the game (Na) affect the probability of winning the prize. For example we can take account of economies and diseconomies of scale by changing from $P = Na/(Na + K)$, to $P = Na^d/ (\Sigma\ Ni^d)$.

i. The payoff function for a typical player now becomes:

- $Ra^e = [Na^d/ (\Sigma\ Ni^d)]Y - Na\ C$

ii. Differentiating with respect to Na now yields:

- $dRa^e/dNa = \{[dNa^{d-1} / (\Sigma\ Ni^d)] - Na^d (dNa^{d-1}) / (\Sigma\ Ni^d)^2 \}Y - C = 0$

iii. To find the symmetric equilibrium, note that Na = Ni for all i = 1, 2, ... N, so:

- $\{[dNa^{d-1} / (NNa)] - Na^{d} (dNa^{d-1}) / (N^2Na^2)\}Y - C = 0$, or putting the numerators over a common denominator and collecting a few terms:
- $\{[dNNa^{2d-1} - dNa^{2d-1})] / (N^2Na^{2d})\}Y - C = 0$, or
- $\{[d(N-1)Na^{2d-1})] / (N^2Na^{2d})\}Y - C = 0$

iv. solving for Na*, yields the individual's number of tickets (level of resources invested in the contest) at the symmetric Nash equilibrium:

- $Na^{**} = [(N-1)/ N^2] (dY/C)$

v. Note that when d=1 and N=2, as above, $Na^{**} = (1/4) (Y/C)$, as before.

- However, the **total expenditure** on "rent seeking" is again NC times this amount, or d(N-1)Y/N.
- Note that total expenditures **will now exceed Y**, whenever d> (N-1)/N.

I. To summarize:

i. The more players are in the game, the less each spends.

ii. However, the total spent rises with the number of players.

iii. In games with constant returns (the classic contest function) the total investment in the contest approaches the value of the prize (Y) as the number of players approaches infinity.

iv. Contests with increasing returns may have "super dissipation," where more resources will be invested in the contest than the prize is worth.

v. (Note that no player will routinely play such games. However, "no one" playing is also not an equilibrium, so potential players may play mixed participation strategies--more on that later in the course.)

J. There are a surprisingly large number of applications of these rent-seeking-lottery games.

- Essentially any contest in which additional resources increases the probability of winning, *or the fraction of the prize that is won*, can be modeled with such functions.
- Indeed, a very large "contest" literature has emerged in the past ten or twenty years that explores such functions.
- To this point, the "Tullock" contest function has been most widely applied to represent interest group politics, although it can be used to represent crime, terrorism, etc. as noted above.
- Note that dissipation--the cost of the "competition"--is an important indicator of social welfare, particularly in contests that are "unproductive" and therefore wholly redistributive.

K. Game theory can also be used to represent less concrete settings.

- For example, payoff fuctions can be represented using abstract functions.
- And, equilibrium strategies can be characterized using a bit of calculus.

L. Illustration: consider a *symmetric game in which each player has the same strategy set and the same payoff function.*

i. Suppose there are just two players in the game, Al and Bob.

- Let the payoff of player A be G1 = g(X1, X2) and that of player B be G2 = g(X2, X1) where X1 is the strategy to be chosen by player 1 and X2 is the strategy chosen by player 2.

ii. *Each player in a Nash game attempts to maximize his payoff, given the strategy chosen by the other.*

- To find payoff maximizing strategy for player A, differentiate his payoff function with respect to X_1 and set the result equal to zero.
- The implicit function theorem implies that his or her best strategy X_1^* is a function of the strategies of the other player X_2, that is that $X_1^* = x_1(X_2)$.
- A similar reaction (or best reply) function can be found for the other player.

iii. At the Nash equilibrium, both reaction curves intersect, so that

$$X_1^{**} = x_1(X_2^{**}) \text{ and } X_2^{**} = x_2(X_1^{**})$$

VII. Review Problems for those Interested in the Mathematical Appendix

A. Let R be the "reward from mutual cooperation," T be the "temptation of defecting from mutual cooperation," S be the "suckers payoff" if a cooperator is exploited by a defector, and P be the "Punishment from mutual defection." Show that in a two person game, relative payoffs of the ordinal ranking T > R > P > S are sufficient to generate a prisoner's dilemma with mutual defection as the Nash equilibrium.

B. Write down an assurance game and assume that the players initially find themselves at the less desirable Nash Equilibrium. Show that your trust problem can be solved by subsidies of various kind. Explain how this game differs from a PD game. Can subsidies also be used to solve a PD game?

C. Suppose that the inverse demand curve for a good is P = 100 - Q and that there are two producers. Acme has a total cost curve equal to C = 5Q and Apex has a total cost curve of C =10 Q. Each firm controls its own output. Prices are determined by their combined production. Characterize the Cournot-Nash equilibrium to this game.

D. Suppose that there are two neighbors, Ms 1 and Ms 2, each of whom enjoy playing their own music loudly enough to annoy the other. Each maximizes a utility function defined over other consumption, C, the volume of their own noise, and that of their neighbor's (a bad). Ms 1's utility function is $U_1 = C_1^{0.5} N_1^{0.5} N_2^{-0.5}$. Ms 2 has a similar utility function and each has a budget constraint of the form , Yi = Ci + Ni.

i. Characterize each neighbor's "best reply" or "reaction" function, and determine its slope.

ii. What happens to neighbor 1's reaction function if his income rises?

iii. Show the effect that a simultaneous increase in each neighbor's income has on the Nash equilibrium of this game.

iv. Is there anything strange about this game?