I. A Short Introduction To This Course

A. The primary aim of this course is to teach you how to model economic choice settings mathematically. Its secondary purpose is to help prepare those who are thinking about graduate school in economics and related fields. The tools and concepts that are focused on are chosen with both aims in mind.

i. This is not a difficult course for students that have mastered calculus, have a good mathematical intuition, understand the geometry of intermediate micro-economics and who have enough discipline to work homework problems on their own.

ii. However, it is a very difficult course for those who do not have a good grasp of calculus and lack the discipline or free time to solve homework problems (only a few of which will be done during class time).

B. The class website (link) is the main source of the class materials.

- The class website provides a web-syllabus (and a pdf syllabus) and approximate timing of all the blocks of materials covered in the class.
- It also includes "due dates" for the 8 homework assignments that are each due roughly a week after a block of material is finished in class. It also provides the exam dates.
- The 8 homework assignments are on the class Ecampus site. No homeworks are accepted after their due date without a note from a doctor or hospital. They are online and so can be done anywhere in the world with internet access and at any time before they are due.
- A few prerecorded class lectures (for my travel dates) will be posted at the class Ecampus site.

C. Your grades are based on the 8 homework assignments, 2 exams, and a final paper, plus a bit of extra credit for helpful class participation.

D. The course is based on the web notes, but the class lectures are not simply me reading my lecture notes.

- The class lectures are shorter somewhat more focused versions of the notes with many more examples.
- There are also more diagrams in the in-class lectures than in the lecture notes, which should help get the intuition behind the math across.
- The lectures are also an excellent time to ask questions. All questions are welcome—and usually benefit most students in the class!
- There will also be a few minutes set aside—normally, once a week, on Thursdays—for in-class practice exercises. These also contribute to your class participation marks.

II. On the Methodology of This Class: Science and the Usefulness of Models

A. Essentially all scientific work attempts to determine what is general about the world. To do so, both individual scientists and teams of scientists use combinations of models, logic, and empirical evidence to determine what is (approximately) true. Science is a "work in progress," although there are theories and facts that most experts believe to be true, every theory and every fact is open to challenge and subsequent revision. It is the process of theorizing, testing, and critiquing theories that gradually generates scientific progress.

i. An example of a facts and theories that most educated persons accept is that, although individual sun rises may be more or less beautiful, ALL sun rises on Earth are caused by the Earth's daily rotation in combination with light generated by its nearest star—the sun. Since Copernicus in the sixteenth century and Newton and Galileo in the seventeenth century, this explanation for our daily cycle of light has been the explanation (theory) accepted by all educated persons.

ii. Similarly, economic theories of decision making imply that all persons take account of different characteristics of an automobile when they purchase a car, but all persons take account of the price of a car. That one's (bounded) personal savings and ability to borrow can also be used to purchase goods other than automobiles implies that the higher is the price of an automobile, the less likely a given consumer is to purchase a particular car, *other things being equal*.

B. Theories not only explain relationships and facts that people "know," but also tend to induce experts (scientists) to look for other relationships and facts implied by a theory, many of which have not previously been known or predicted. A good theory explains previously unknown or ignored facts in addition to well-known facts.

i. However, no theories can explain all the facts or relationships that they might be said to have implications about.

- For example, consumers who believe that markets are "efficient" may use price as a proxy for quality and tend to buy more expensive autos over less expensive autos because they believe that the more expensive cars are of higher quality.
- This tends to be true if other consumers are well-informed and use similar gauges of quality.

ii. Quality differences also matter. Demand curves for single goods may "slope downward" but higher priced goods may be purchased more frequently than less expensive substitutes, because they are regarded to be of higher quality (as true of Apple phones relative to many far cheaper Android phones in the US).

iii. Note that including this property of market prices can be explained, but doing so requires a more general or encompassing theory than the simpler one taught in principles of economics classes and used in most mathematical models of consumer choice.

C. This process of finding relationships or general rules that explain "how the world works" is a complex and time-consuming process. It can be described as a joint exercise in logic (model building), observation (empirical testing), and reflection about the limits of the model's assumptions. It is a nearly endless cycle of model building (hypothesizing), empirical testing, refinement, retesting, and so, forth (a process that I sometimes call boot strapping).

- Through this process our understanding of the world gradually improves and is in a sense perfected.
- Even after several thousand years of efforts to understand the universe, some aspects of the world and universe are understood far better than others.
- Some models are, thus, better than others—more precise, more encompassing, or more useful.
- However, even somewhat imprecise models (mathematical representation of a principle or relationship) are often very useful.

III. Mathematical Model Building in Economics

A. Models are useful, because the world is complicated—at least on the surface—and models can often help identify regularities that are not obvious at first sight.

  i. For example, price theory allows one to understand some of the underlying causes of market prices as an outcome of independent choices made by firms and consumers, but which none the less tend to cluster in relatively small price ranges for each good in particular places.

  ii. The models used to do so, strip away a lot of the complexity of the actual decisions to purchase and produce goods in order to focus on factors that are common to all such choices.

  iii. Physics does something similar when it uses the idea of gravity to explain the course of a leaf falling to the ground or the orbits of planets.

  iv. Gravity is common to all such movements, although other factors such as wind, shape, or entropy may also affect the course of an object. Neglected factors, such as the wind and aerodynamics, also matter. The effects of gravity are more universal—although not always fully determinative.

B. "Model building" always starts with ideas about what is truly important or commonplace in the relationships of interest (hypotheses).

C. Once a model has been deemed useful or convincing because it provides plausible explanations and has been affirmed by statistical and other empirical tests, it may be considered to be "true" for most purposes and "standard" forms of the model tend to emerge.

  i. For the most part, this course focuses on such "standard" models—although as mentioned above, even "standard" or "mainstream" models may have failed some tests and may be improved through time.

  ii. Most of the standard graphical models taught in principles of economics and intermediate micro-economics have mathematical foundations, because the math was often worked out before the geometry.

  - Utility functions are mathematical functions that represent utility as functions of goods purchased and consumed or activities undertaken. Production functions are mathematical functions that represent outputs as functions of inputs.

- General equilibrium models were first presented in intuitive ways (by Walras), but they were in a sense simplified and subsequently worked out mathematically. They and many other economic theories are often clearer in their mathematical expression than in their geometric ones.

- Game theoretic notions of best-reply functions or reaction functions were first worked out mathematically, and are in many cases somewhat clearer in their mathematical than in their graphical representations.

iii. This is not to say that geometric models are a waste of time. In many cases our geometric intuition is stronger than our algebraic or calculus intuitions, and so we can "squeeze" more insights from a geometric representation than from equations. A lot of graphs will be used in this course to illustrate what is going on in the calculus.

- Moreover, there are some economic ideas that are far easier to explain geometrically with a diagram than with other mathematical tools.

iv. It also bears noting that the empirical part of economics (econometrics) also has mathematical foundations. Statistical theory was developed using mathematics along with a few assumptions about probability distributions. Econometrics uses various statistical methods to "test" hypotheses generated from economic models.

v. Econometric tools all have mathematical foundations.

- Moreover, nearly all statistical tests assume that you know the right functional form of the models that you estimate and also know the relevant error distributions (normal, etc.).

- Not all persons who use econometrics write down the economic models that generate the prediction being tested, and they nearly all implicitly make mathematical assumptions about the relationships of interest. They do so whenever they use linear or log-linear representations as the basis for their estimates (as in regression analysis and generalized least squares).

D. The use of mathematics for analyzing markets has been undertaken for less than two hundred years. So, in a sense, it is a relatively new approach to understanding how markets operate.

- Trade, production for sale, and extended networks of exchange (markets) are themselves thousands of years old.

- The logic of supply and demand in their own markets would have been "intuitive" to many firm owners for hundreds or thousands of years before the models were worked out in the nineteenth century.

- **What the mathematical and geometric models show is that intuitions worked out about a particular product or market generalize to essentially all markets, not just for the products sold by a particular merchant or industry.**

- There are commonplace relationships that exist in all markets, and these have commonplace implications about both existing prices, how they change through time, and their effects on behavior.

- Competition among sellers for the purchases of consumers also have commonplace relationships and implications for both prices, the manner in which goods are produced, input prices and so forth.
- Mathematically showing this is the "main dish" of this course.

E. Not all economic models are mathematical, but **mathematical models have many advantages over intuitive modeling methods.**

- Perhaps the most important is LOGICAL CONSISTENCY. (What you have derived is true, *given your assumptions*.)
- Another important feature is that the mechanics of math allow one to push the logic of a model to places where your intuition stops functioning or becomes confused.
- Another is that economic models usually make clear assumptions about the choice and market setting being analyzed. This allows the range in which a particular conclusion applies to be better understood.
- (Not all these assumptions are immediately obvious, and in some cases economics advances by better understanding what one's or previous generations of economic assumptions are.)

F. Mathematical models allow mathematics (deduction) to be used as an "engine of analysis."

i. Those deductions often reach surprising and/or counter intuitive conclusions that can be subjected to empirical tests.

ii. If the model is "close enough" to reality, then conclusions reached from the models will also be properties of the real world. (If not, then not.)

G. Economists normally attempt to deduce properties and implications of purposeful decision making using "**lean models**," that is to say models that make very few assumptions about decision making in the economic settings of interest to economists in general to a particular the model builder.

i. Typically, assumptions are made about individual purposes—namely that they can be represented as net benefit or utility functions—at least within the economic choice setting of interest.

ii. In addition, choice settings are generally characterized with a feasible set of some kind--as with a budget constraint, budget line, or budget set for consumers or a technologically feasible production possibility set or function for firms.

iii. As will be seen in this course, the "design" of economic models is partly based on economic intuitions, partly on mathematical tractability, and partly on conventions (other folks have used the same methods).

iv. Mathematical models often produce results that are "intuitively plausible." Their rigor (mathematical methodology) shows that the results are not just an intuitive gut reaction. It demonstrates that a plausible relationship or principle emerges logically from the behavior of rational decision makers in commonplace circumstances.

IV. Mathematical Individualism

A. Essentially all micro-economic models rely upon an assumption or approach referred to as **methodological individualism**.

i. Methodological individualism is an approach to social phenomena that assumes that all social phenomena—including markets--can be analyzed in terms of the individuals in the networks, groups, or societies of interest, and the external constraints they confront (budgets, laws, technology, strategy sets, etc.).

ii. In other words, "networks," "groups" and "societies" are all composed of individuals making independent decisions in particular choice settings. The properties of "networks," "groups," and "societies" emerge from the choices of the individuals in those networks, groups, and societies.

iii. The types of decisions reached by individuals are partly determined by their own circumstances, but are also influenced by various rules and norms for group and social membership.

- When one participates in market networks, one does not steal nor use extortion to gain what he or she wants—rather one engages in various forms of voluntary exchange.
- That is one of the fundamental "rules" or constraining principle of market networks.

B. Micro-economics thus begins with individuals, rather than groups or societies.

- To model individuals, "we" normally assume that individuals have goals that can be characterized mathematically with utility functions.
- It is also assumed that their circumstances (opportunities) can also be characterized mathematically (as with budget sets or constraints).

C. Most economic models assume that individuals are rational in the sense that they have purposes or aims that they attempt to advance through their actions and are not too often wrong about the best way to advance those purposes.

i. Examples include:

- Rational individuals may be "net-benefit" maximizers.

Given $B=b(Q)$ and $C= c(Q)$, we assume that "Al" chooses Q to maximize $N = B-C$.

This requires $dN/dQ = dB/dQ - dC/dQ = 0$

- Rational individuals may be "utility" maximizers.

Given $U=u(a, b)$ where $W=a+b$, we model "Bob's choice as finding the combination of a and b that maximize U for given W.

- Rational individuals may also simply be modelled as people who simply make consistent choices.

If Al prefers A to B and B to C, then Al prefer A to C

ii. All three of these models of individual decisionmaking are used by economists because they allow choices by rational individuals to be represented mathematically (with sets and equations) as well as with diagrams. This allows mathematics to be used to understand and predict the consequences of changes in circumstances on individual and group choices and on subsequent market outcomes (comparative statics).

iii. The models used by economists are all very similar to one another in spirit, in that they nearly all assume that people are basically rational—at least in economically relevant circumstances. What differs is how they model rationality, circumstances, and their effects on choices.

- There are subtle changes in the assumptions that one makes as one shifts from: consistent preference orderings to utility maximizing to net-benefit maximizing models of choice, which we'll be reviewing during the course, although these will not be the course's main focus.

iv. The utility maximizing models are the most widely used of the three, but the others are also widely used, the former in mathematical economics and the latter in undergraduate course and applied work.

- In addition, there are extensions of these models to take account of specific circumstances.
- A person that owns a firm may be modeled as a "profit maximizer" or a "utility maximizer" that takes price in their work and makes tradeoffs between profits and the quality of their products.
- A person that is running for an elective office in a democracy may be modeled as a "vote maximizer" or as a person that wants to obtain sufficient votes to win the election, but is otherwise interested in advancing his or her ideas about a "good" society.
- In setting where risk or time are important, the persons may be modeled as maximizing "expected" utility or net benefits, or the "present value" of utility flows or net benefits.
- The course will show how such choices can all be modelled using mathematics.

C. Nearly all models that can be represented with equations can also be represented with diagrams and vice versa.

D. However, this course will focus for the most part on mathematical representations using basic results and tools from calculus.

i. Calculus allows a lot of fairly general results to be worked out, without making too many outrageous assumptions.

ii. Many of the results are much more general than can be shown with diagrams, and in some cases more intuitive (once you used to thinking about choices and choice settings using calculus).

- As noted above, many of our geometric representations were first worked out mathematically and make sense because of their mathematical foundations.
- Although, it also bears noting that there are cases in which the geometry was worked out first and provides the clearest indication of the usefulness of particular models and results.

Appendix I: Axiomatic Choice: Some Fundamental Concepts and Definitions from Mathematics used in Consumer Theory [*Optional Material, mainly for students heading to graduate school in economics*]

A. When the rules of logic are applied to numbers the result is mathematics.

i. Most of the mathematics we have been taught can be deduced from a few fundamental assumptions using the laws of logic.

- (See the postulates of Peano, an Italian mathematician (1850 - 1932).)

ii. Some mathematical economists are very attracted to the axiomatic approach (See Debreu's book for an early example), and we will spend a bit of time today seeing who that approach can be used to model human choices.

B. Some fundamental properties of logical and mathematical **relationships**:

i. DEF: Relationship R is **reflexive** in set X, if and only if aRa whenever a is an element of X.

ii. DEF: Relationship R is **symmetric** in set X if and only if aRb then bRa whenever a and b are elements of set X.

iii. DEF: Relationship R is **transitive** in set X if and only if aRb and bRc then aRc when a, b, and c are elements of set X.

iv. Recall that within the set of real numbers, there are relationships which are symmetric (equality),reflexive (equality) and transitive (equality, greater than, less than, greater than or equal than, less than or equal than).

v. In economics there are also several relationships which possess all three properties, and some that exhibit only transitivity.

vi. In general, economists assume that "rational" preference orderings satisfy all three of these properties. Strong and weak preference orderings are transitive, while indifference is transitive, symmetric and reflexive.

- Indeed, rationality in microeconomics is often defined as transitive preferences.

C. Note that the indifference relationship, I, can be defined in terms of the weak preference relationship R.

a. The weak preference relationship R means "at least as good as."
b. Note that if aRc and cRa, then aIc.

D. Similarly, the strong preference relationship, P, can be defined in terms of the weak preference relationship.

c. The strong preference relationship means "better than."
d. Note that if aRb but b~Ra then aPb.

E. DEF: A **function** from set X to (or into) set Y is a rule which assigns to each x in X a unique element, f(x), in Y. Set X is called the domain of function f and set Y its range.

- On most diagrams from math classes, the domain is the horizontal axis (X) and the range is the vertical axis (Y).
- However, most textbook diagram in economics have the domain of the function on the vertical axis (P) and the range of the function on the horizontal axis (Q). For example demand functions go from P (prices) into Q (quantities a consumer is prepared to purchase).
- This is evidently Marshall's fault, who decided that the diagrams were easier to draw this way—possibly because of the use of blackboards back in the 1890s.

F. DEF: A **utility function** is a function from set X into the real numbers such that iff aPb then U(a) > U(b) and if aIc then U(a)=U(c) for elements of the set X.

Note that a utility function defined in this way does not characterize satisfaction, but rather choices based on transitive weak preferences.

However, if you think of utility as "satisfaction" or closeness to some "goal" you intuition will be correct.

In diagrams, indifference curves are simply graphs (plots) of all combinations of two goods that generate the same utility.

G. Note that the assumption that a utility function exists, is equivalent to the assumption that individual preferences are transitive.

- Real numbers are transitive with respect to equality (indifference) and greater than (strict preference).
- The assumption that all combinations of the goods can be evaluated with the function implies that preferences are **complete**: each bundle (combination of goods and/or "bads") has a unique rank. a. Every bundle of goods generates either more or less or the same utility level as other goods.
- (Some theorists make a distinction between complete and incomplete utility mappings from X to R, but this distinction is not important for "routine" decisions.. Why?)

H. **Some important definitions and concepts from Set Theory**.

i. DEF: An infinite series, $x_1, x_2, \ldots x_n$ is said to have a **limit** at x* whenever for any d >0, the interval x* d, x* + d contains an infinite number of points from the series. (That is to say, x* is a limit point of a series in any case where there are an infinite number of elements of the series arbitrarily close to x*.)

ii. DEF: A set is **closed** if it contains all of its limit points.

iii. Def: A set is **bounded** if every point in A is less than some finite distance, D, from other elements of A.

iv. Def: A set is **compact** if it is closed and bounded.

- Most opportunity sets in economics are assumed to be closed and bounded.
- Def: A set is **convex** if for any elements X_1 and X_2 contained in the set, the point described as $(1-d)X_1 + dX_2$ is also a member of the set, where $0<d<1$.

v. Essentially a convex set includes all the points directly between any two points in the set.

- That is to say, any convex (linear) combination of two points from the set will also be a point in the set.
- Thus, a solid circle, sphere, or square shaped set is a convex set but not a V-shaped or U-shaped set.
- To see this, draw one of these figures, pick serve representative pairs of points and connect them with a line interval (cord). The line interval will lie entirely in the set.
- What other common geometric forms are convex?
- (Why does a series of convex combinations trace out a cord? Because as d varies from 0 to 1, the "point" characterize moved along a line from one point (X_1) to the other (X_2.)

vi. Example from economics: "better sets" are usually assumed to be convex sets. That is to say, the set of all bundles which are deemed better than bundle a is generally assumed to be a convex set.

- Another example is the budget set, the set of all affordable commodities give a fixed wealth and fixed prices for all goods that might be purchased.

vii. **Convexity and compactness assumptions are widely used calculus and graphical models** of human decisionmaking, because they make smooth continuous functions (and lines) possible.

- Opportunity sets and production possibility sets are nearly always assumed to be convex and compact.

Next chapter: the shapes of functions and optimization