R. Congleton

Chapter 2: Rationality, Optimization, and Shapes of Functions

I. Introduction: Modelling Individual Decisions

A. As discussed in the Chapter 1, economics and the "rational choice" strands of other social sciences use "methodological individualism" as the conceptual foundation for their theories and models. This means that social phenomena such as markets and political systems are regarded to be joint consequences of individual decisions and actions.

i. Thus, modelling social phenomena such as market networks begins with modelling individuals in various contexts—chiefly in their roles as consumers, input providers, investors, and owners of firms and other assets.

ii. Individual decisions to participate in markets are made one at a time, and the great networks of exchange, production, and innovation that make up commercial societies are consequences of the decisions of many, many, independent decisions—coordinated by market prices, civil and criminal laws, and internalized norms.

iii. Social phenomena do not emerge from a single decision but are consequences of the decisions of many individuals.

iv. For the most part, economics assumes that well-functioning markets have already emerged (as they had by the nineteenth century when neoclassical economics emerged) and attempts to understand how those markets operate—given that they emerged from the decisions of dozens, hundreds, thousands, millions or even billions of independent decisions by individuals and relatively small groups of individuals (firm owners).

v. Thus, micro-economic analysis begins with the analysis of individual consumer and firm decisions.

B. Economists generally assume that individuals are rational in the sense that they "optimize." Optimization implies that individuals try to determine the "best way" (the optimal way) to pursue their interests.

vi. Rational individuals use their resources (wealth, time, health, talent) to advance their own personal objectives as well as they know how.

i. In consumer theory, the consumers maximize utility given their budget constraints.

ii. In the theory of the firm, firms are assumed to maximize profit, given production technology, input prices, and market opportunities.

iii. In rational choice based political theory, candidates are assumed to maximize votes given the positions of other candidates and the preferences of voters.

iv. All these are settings where individuals optimize, given various constraints on the actions that might be undertaken. Not all actions are feasible, but often a very large number are.

C. That individuals "optimize" has a surprising range of implications for markets and other social activities in which individuals participate.

- Neoclassical economics is largely the result of more than a century of attempting to work out those implications.

D. The optimizing model of humankind can be modeled or characterized with the mathematics of constrained optimization. That bit of mathematics provides the core of this course and is the main focus of the next chapter. This chapter focuses most of its attention on assumptions that lie behind the mathematics of optimization.

i. The assumptions include the "rationality" of individual decision making in the sense of internal consistency and efforts to optimize.

ii. They also include assumption about the aims of individual (utility, income, profits) and the choice settings confronted (opportunity sets).

E. To make the mathematics tractable and useful, a variety of **assumptions** about preferences and constraints are made.

i. For example, it is usually assumed that individuals control things or action that are infinitesimally variable.

- In principle, it assumes that individuals can allocate .0001 seconds to an activity or purchase .00001 ounces or grams of something at grocery stores.
- One may not actually be able to do that in reality, but often little damage is done by that assumption.
- And, that assumption allows calculus and algebra to be used to build models of individual choice and market equilibria.

ii. When there are constraints—for the most part they are assumed to take the form of **convex sets.**

- Convex sets are collections of points in which every line connecting two points in the set (a cord) includes only points that are also points in that set.
- For example, the points inside a circle are a convex set, as are the points along a straight line.
- A budget set, which tend to be triangular, is also a convex set.
- A set that has a doughnut-shape or a figure-eight shape is **not convex**, because there are lines connecting two points within those sets that includes points outside those sets. (If this is not obvious to you, take a minute to draw such sets and see.)
- Similarly, the set of points that form the letters "A," "B," "C," and most other letters are **not convex**, because if you connect points from one side to the other they often include points that are not part of those letters.

iii. It is also normally assumed that the objective functions (utility, profits, output, etc.) are continuous function, strictly concave, and twice differentiable.

- This assures that first and second derivatives can be calculated or at least characterized.
- The first derivatives characterize concepts such as marginal benefit and marginal cost.
- The second derivatives determine whether the marginal benefit curve is upward (or downward) sloping and whether the marginal cost curve is upward sloping (or downward sloping).

II. Three Types of Concavity

A. **A function** is a mapping from one set (often Q or quanity in economics) into another set (such as net benefits, costs, benefits, utility, profits, revenue, etc.)

B. Many of the mathematical properties of a given function can be deduced from its "shape."

C. One of the most widely used characterizations of a function's shape in economics is *concavity*.

D. There are **three notions** of concavity used in economics, although in this course, only the first and second are used outside of this chapter.

i. DEF: **Strictly Concave**: function f is *strictly* concave iff

$$\alpha f(X_1) + (1-\alpha)f(X_2) < f(\alpha X_1 + (1-\alpha)X_2) \qquad \text{where } 0 < \alpha < 1.$$

- Geometrically this means a function is strictly concave "if and only if" all the points on a line segment connecting any two points on a function always lies "beneath" the function of interest.
- **Strict Concavity is the assumption about functions that is most often used in this course.**
- A strictly concave function has at most one maximum, which allows us to characterize choices that are very specific—whether the choices are unconstrained or constrained by some convex set.

ii. DEF: Concavity: function f is concave iff

$$\alpha f(X_1) + (1-\alpha)f(X_2) \le f(\alpha X_1 + (1-\alpha)X_2) \qquad \text{where } 0 < \alpha < 1.$$

- Geometrically this means that points on a cord (line segment) connecting any two points of a function always lies on or beneath the function of interest.
- A straight line is concave, but not strictly concave.
- Other strictly concave function are also concave.

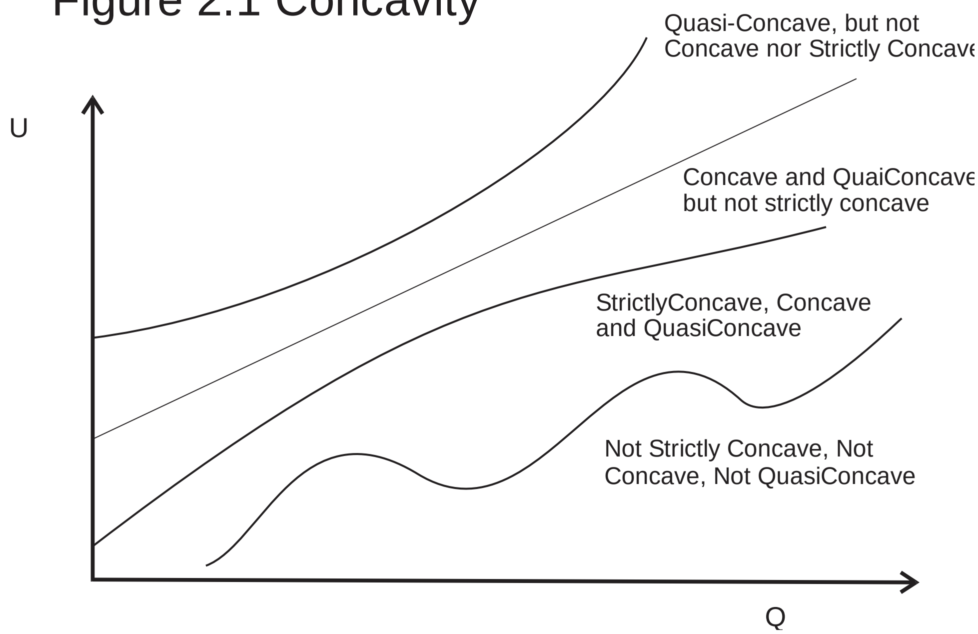
iii. DEF: Quasi-Concave: Concavity: function f is *quasi* concave iff

$$f(X_1) < f(\alpha X_1 + (1-\alpha)X_2)) \quad \text{where:} \quad \text{and } f(X_1) < f(X_2) \text{ and } 0 < \alpha < 1.$$

- The values of a quasi-concave function always lies above the lower of the two end points of a cord connecting any two points on the function.

- Any monotone increasing function is quasi-concave, but it is not necessarily concave or strictly concave, because it may increase at an increasing rate.

Figure 2.1 Concavity



III. Maxima and Minima of Functions

A. Strictly concave functions have a number of useful properties in the context of "optimizing" behavior.

i. A *strictly* concave function has at most one maximum. (Draw some pictures to see why.)

ii. However, a **concave function** may have an infinite number of global maxima, but if there is more than one maximum, they make up a continuous linear interval. (A horizontal line is concave, but not strictly concave.)

B. DEF: The **global maximum** of a function, f(x), is a value, f(x*), that exceeds all others over the entire range of the function (e. g. for every neighborhood of x*).

C. DEF: A **local maximum** of a function, f(x), has a value which exceeds those of other points within a finite neighborhood of x*. That is, f(x*) is a local maximum if $f(x^*+e) < f(x^*)$ and $f(x^*-e) < f(x^*)$ for $0<e<E$, for some $E>0$.

- Note that if a function has a global maximum, then that global maximum is also a local maximum.

- However, because a function may have many local maxima, only one of those can be a global maximum.

D. Derivatives of functions can be used to characterize *sufficient* conditions for concavity, strict concavity, and therefore also for global maxima and minima.

i. A function is *strictly concave* if its first derivative(s) is positive, and its second derivative(s) is negative over its entire domain.

ii. A function is concave if its first derivative is positive, and its second derivative is less than zero over its entire domain.

E. Functions may have local and global maxima, although most of the functions used in economic model-building are assumed to be strictly concave and so have at most one maximum (e.g. only one local maximum, which is also its global maximum).

i. A function is at local maximum at point Q* if and only if (iff) its first derivative at Q* has the value zero and its second derivative is negative within a finite neighborhood around Q*.

ii. A point, Q*, is the global maximum of function f(Q) if its first derivative has the value zero at Q* and its second derivative is negative throughout the domain of the function. (Notice that in this case function f(Q) is strictly concave.)

iii. Maxima are, as it turns out, important for constrained optimization.

- With a particular domain, as with $0<Q<2$, any function, f(Q) will have a highest value.
  - This would be the constrained optima or maximum for function f(Q) within the domain from 0 to 2. It would be the "constrained" optimum.
  - Note that there may be more than one such optima, as when f(Q) is a horizontal straight line or a simple sine curve.
- However, at least one maximum will always exist. This is simply a property of real numbers, within any set there will always be a largest value (number).

iv. When the function is strictly concave and the constraint is a convex set (as with the interval example above, $0<Q<2$) there will be a unique maximum (optimum).

v. Thus, when a consumer has a constraint set that is convex (e.g. choose a "bundle" within a particular convex set such as budget set), and attempts to maximize a strictly concave objective function such as a utility function or net benefit function, the unique maximum will be the rational consumer's choice.

- Virtually all of the models of consumer choice examined in this course are grounded in this very general property.
- It is used, for example, to derive demand and supply curves from consumer and firm choices.

vi. What we'll mainly be doing in class and in the lecture notes is modeling a wide range of choice settings where the optimization model of rationality yields interesting and useful predictions about individual, firm, and market behavior.

IV. Unconstrained and Constrained Optimization

A. Several of the choice settings interest to economists can be regarded as instances of "unconstrained" optimization, because there are no bounds placed on the control variables to be determined by the decision maker's choice—or if there are bounds they are relatively unimportant (non-binding).

i. Firms, for example, are usually assumed to be able to pick any Q they want to when they choose a production level to maximize profits.

ii. Consumers can pick any Q they want to maximize net benefits (consumer surplus).

iii. Such choices can be easily modeled using calculus.

- A firm's choice can be regarded as that which sets marginal profits equal to zero, which requires marginal revenue to equal marginal cost, **because** maximizing $\Pi = R(Q) - C(Q)$ requires setting $d\Pi/dQ = 0$, which requires $dR/dQ - dC/dQ = 0$.

- Similarly, a consumer that maximizes consumer surplus or net benefits (N) requires a quantity Q that sets marginal benefit to equal marginal cost, **because** maximizing $N = B(Q) - C(Q)$ requires setting $dN/dQ = 0$, which requires $dB/dQ - dC/dQ = 0$.

iv. Note that both these examples assume that the objective functions Π and N are assumed to be **strictly concave and twice differentiable**.

- Both the profit and net benefit functions are assumed to have positive first derivatives in Q and negative second derivatives.

B. However, many other economic choice settings involve constraints of one kind or another.

i. For example, utility maximizing individuals are constrained by a "binding" budget constraint.

- When the consumer is choosing how much of goods X_1 and X_2 to purchase, his or her choice of X_1 affects how much X_2 can be purchased.

ii. In such cases, the above "unconstrained" maximizing (or minimizing) technique *cannot* be directly applied by either the individuals themselves or those attempting to model their behavior.

- However, it is often possible to "substitute" the constraint(s) into the objective function in a manner that allows the above maximization technique to be used even when there are constraints.

- Alternatively, one can use the Lagrangian technique (which will be developed in the next chapter).

- [We will use which ever method is "easiest" when modeling the choice settings of interest in the course. This will often be the substitution method.]

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V. A Digression: Linking the Geometry and Calculus of the Net-Benefit Maximizing Models

A. In principle and intermediate micro-economics textbooks, the authors often use two letters to characterize a function rather than just one letter as in most of mathematics. This true of most geometric representation—e.g. when drawing "curves" or "lines" to illustrate the logic of some particular feature of choices in economic settings This is to help you remember what each term is.

- However, **this will not be routinely done in this course**, although it will be done in this chapter to help students see the links between the geometry and particular mathematical ideas, and it will sometimes be done when introducing terms during the first part of the course.
- Total benefits (TB) will mostly be written as B, Total cost (TC) will mostly be written as C, and total revenue (TR) will mostly be written as R.
- Instead of marginal cost (MC), we will use either dC/dQ, or C'(Q), or C_Q –which is to say we'll be using the notational conventions of calculus and economic analysis that employ calculus.

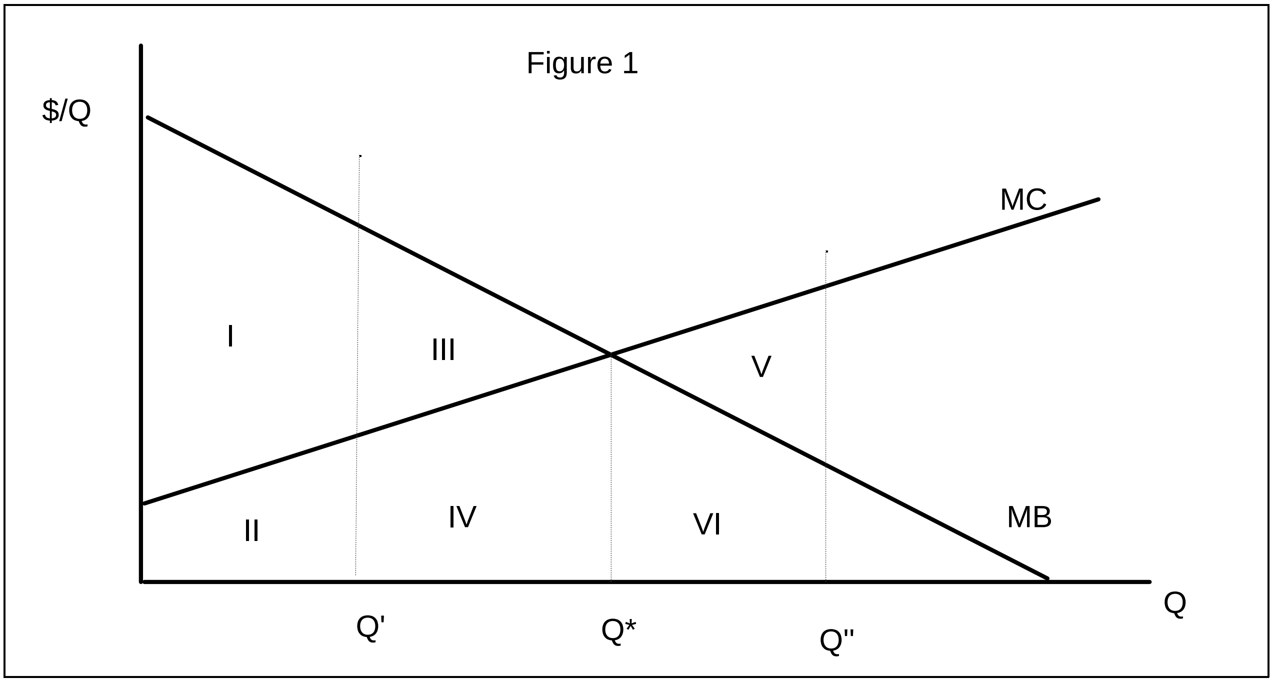
B. The change in benefits, costs, etc. with respect to quantity consumed or produced is generally called Marginal benefit, Marginal cost, etc. In this course, this is simply the derivative of the relevant total function with respect to Q.

- This idea also can be applied to cost and benefit functions that are not "differentiable" because only discrete units of a good are possible. A more general definition is the following.

C. DEF: **Marginal "X"** is the change in Total "X" caused by a one unit change in quantity. In cases in which the domain of X is continuous rather than discrete, it is the slope of the Total "X" curve.

"X" ε {cost, benefit, profit, product, utility, revenue, etc.}

- *Important Geometric Property*: Total "X" can be calculated from a Marginal "X" curve by finding the area under the Marginal l "X" curve over the range of interest (often from 0 to some quantity Q). This property allows us to determine consumer surplus and/or profit from a diagram of marginal cost and marginal revenue curves.
- In terms of calculus, this is the integral of the first derivative of the relevant benefit, cost, utility, or production function, and returns the original function, except for an unknown constant, which often is 0 in economic applications.
- (An exception to that rule, as well see later in the course, is the cost function, where the unknown constant is total fixed cost, which may be relevant for calculating short run profits.)



D. Using Marginal Benefits and Costs to Calculate Total Benefits and Costs

i. Given the marginal cost and marginal benefit curves in Figure 1, it is possible to calculate the total cost of Q' and the total benefit of Q' .

ii. These can be represented geometrically as areas under the curves of interest. TC(Q') = area II; TB(Q') = areas I + II .

- Note that calculating these areas is equivalent to finding the **definite integral** of the MC or MB functions over the range of interest (about which we will say more later in the course).

iii. One **can calculate the net benefits of any quantity of the good** by finding total benefit and total cost for that quantity, and subtracting TC from TB at the quantity.

- Thus, the net benefit of output Q' is B(Q') - C(Q') = area [I + II] - area [II] = I.
- This is equivalent to finding the integral of B'(Q) – C'(Q) for the range of interest, in this case from 0 to Q'.

Exercise/Puzzle.

- Use Figure 1 to determine the areas that correspond to the total benefit, cost and net benefit at outputs Q', Q* and Q''.

Answers:

- TB(Q') = I+II and TC(Q') = II, so NB(Q') = (I+II)-(II) = I
- (Recall that NB(Q') is sometimes called the consumer surplus from Q' units of the good.)

- TB(Q*) = I + II + III + IV , TC(Q*) = II + IV , so NB(Q*) = I + III
- TB(Q”) = I + II + III + IV + VI , TC (Q”) = II + IV + V + VI , so NB(Q”) = I + III - V

E. If one attempts to maximize net benefits, it turns out that generally one will want to consume or produce at the point where marginal cost equals marginal benefit (at least in cases where Q is very divisible). Note that this is demonstrated in the figure above.

i. In the example above, note that NB(Q*) > NB(Q') and NB(Q*) > NB(Q").

- This is true for any quantity Q’ less than Q* and any quantity Q” above Q*
- Thus, net benefits are maximized at the quantity, usually denoted as Q*, where MB=MC.
- When Q* is not possible for some reason, the same calculations can allow one to characterize choice between particular values of Q’ and Q”
- Note, however, that sometimes one’s best choice does not always equate MB and MC. For example, a very common choice for most consumers is to choose Q* = 0. (How many pink Cadillac’s do you own?) Show that MB doesn’t “usually” equal MC in this case.

ii. In the usual case modeled, a net-benefit maximizing decision maker chooses a consumption level (Q*) such that their own marginal costs equal their own marginal benefits. They do this not because they care about "margins" but because **this is how they maximize their own net benefits**.

- This characterization of net benefit maximizing decisions is quite general, and it can be used to model the behavior of both firm owners and consumers in a wide range of circumstances.
- Moreover, the same calculus and geometry can be used to characterize demand and supply curves, as shown in the next chapter.

Appendix: (Optional, mainly for students thinking about graduate school) Other Families of Functional Forms Sometimes Used by Economists: Cobb-Douglas, Homogeneous, and Homothetic Functions

A. We will not often use the vocabulary and definitions in this section. It is included for students planning on or interested in graduate school in economics.

B. For illustrating examples later in the course, we’ll be using explicit functional forms that are exponential in form, some of which are homogeneous of degree 1, but not all of them.

C. Concrete or explicit functional forms are used in this class and in some economic research because

(a) the results are easier to interpret and so serve as useful illustrations of more general models.

(b) They are also used because they provide a theoretical reason for particular estimation methods.

i. For example, a utility function might be assumed to have an exponential form, $U = aX^bY^c$, where: x and y are quantities of two goods.

ii. Cobb-Douglas functions are special cases of exponential forms where the exponents add up to one, as with b + c = 1.

iii. Cobb Douglas functions are "homogeneous" of degree 1.

D. DEF. A function is said to be *homogeneous of degree k*, if and only if whenever

$$Y = f(X) \text{ , then } f(\beta X) = \beta^k Y$$

i. A production function that is homogenous of degree 1 exhibits constant returns to scale. Doubling all inputs, exactly doubles outputs.

ii. Cobb-Douglas functions and linear functions through the origin (Y = ax) are homogeneous of degree 1.

iii. Occasionally, utility and production functions are assumed to be *homothetic*, a somewhat more general family of functions than homogeneous functions.

E. **DEF**. A **homothetic function** is a composite function of the form H = h(Q(a,b)) where Q is a homogeneous function and dH/dQ > 0 over the entire domain of h or dH/DQ < 0 of the entire domain of h. (E.g. a homothetic function is a monotone increasing or decreasing transformation of a homogeneous function.)

i. Not all homothetic functions are homogeneous.

ii. Homothetic utility functions have linear income expansion paths.

iii. Similarly, homothetic production functions have linear output expansion paths. (*The slopes of the isoquants are the same along any straight line through the origin*.)

iv. Assumptions of homogeneity and homotheticity make models and their implication less general than they would have been with assumptions of monotonicity, concavity, and strict concavity, but the clarity of the results is often felt to warrant such assumptions.

Next chapter: The Calculus of Optimization and economic relationships such as supply and demand and market equilibrium.