I. Rationality as Optimization

A. This chapter shows how the mathematics of optimization can be used to model a variety of decisions in a variety of economic choice settings and to use those models to predict how individuals and markets tend to behave in those settings.

i. It also characterizes three sets of mathematical tools for finding "optimal" values of a "control variable." The tools dealing with constrained optimization are not used much in this chapter, but are introduced and used to characterize a consumer choice, firm choice, and market equilibrium in chapter 4.

ii. **Together these make this chapter and the next the most important chapters in the course**—they provide the mathematical foundations for most of the rest. If you master this chapter and the next one, you'll be in good shape for the rest of the course. And if not, you'll have trouble with the rest, although you'll have a good deal of additional practice in using the same or similar mathematic to analyze other choice settings during the rest of the course—which should help you to master them.

B. The optimization model of rationality allows one to use calculus to model the choices made by individuals in virtually any circumstance, because essentially every choice setting involves optimizing: recognizing what is important (the objective) and what is possible (the constraints). What models generally do is attempt to isolate the essential features of a person's objectives and constraints.

iii. Objectives are normally characterized as functions that map actions of various kinds into numbers that are assumed to be correlated with one's objectives. In some cases, the objective may actually be a number such as profits or outputs of a good to be sold in a market, but in other cases the objective may be subjective and so not objectively measurable, as with utility. However, as long as one can reasonably assume that such unmeasurable objectives can be better advanced by some actions rather than others, we can use numbers (utility values in "utils") to represent (model) the objectives sought (utility, happiness, good character, etc.) and the choices made.

iv. The constraints are also represented in numerical terms. Again, some constraints such as a budget constraint are numerical. One only has so much money to spend or so large of a credit line. But others are not measurable or entirely measurable, such as the extent of one's knowledge or ignorance. But, again if the terms more and less are meaningful (as with more ignorant or less ignorant), we can again represent such constraints using mappings into real numbers.

v. A good deal of the progress in neoclassical economics has involved better representations of objectives and constraints.

vi. In addition, as the scope of optimization has become better recognized, the scope of neoclassical economics has been extended to provide insights into other fields of social and biological science.

- Rational choice models can be used to characterize choices in a very broad range of circumstances, and several Nobel Prizes in economics have been awarded to the economists who first figured out how to do so, as with Gary Becker with respect to many areas of sociology (1992) and James Buchanan (1986) with respect to many areas of political science.

- Other Nobel prizes have been awarded to economists who have argued or demonstrated that there are limits to what rational choice models can explain as with Herbert Simon (1978) and Daniel Kahneman (2002).

vii. In this class we will be exploring the predictable consequences of rational choice models based on optimization. These models can explain a substantial amount of the behavior that we observe in markets and in other walks of life.

C. There are more or less "abstract" and more or less "general" rational choice models that economists have worked out. We'll begin the course with relatively "concrete" models of human objectives and constraints and end the course by showing how similar results often can be found using more general models and mathematics.

- For example, an abstract or general model of utility is an equation like: $U = u(a, b, c, d, e)$ with function u having positive first derivatives for the control variables a, b, c, d, and e, negative second derivatives, and cross partial derivatives equal to or greater than zero. (These assumptions, as we'll see later in the course assure that function u is strictly concave.)

- "Concrete" or "explicit" functional forms of utility functions include equations such as $U = a^{.25} + b^{.15} + c^{.15} + d^{.25} + e^{.125}$ or $U = a^x b^y c^z$ with x, y, and z greater than 0 but with a sum that is less than 1. (Note that both these utility functions are "special cases" of the general utility function above, and both are also strictly concave.)

i. The advantage of "concrete" functional forms is that it is often possible to solve for specific results and thereby make specific predictions. Some of these can be directly tested using econometrics (the portion of statistics used by economists). Their disadvantage is that, as special cases, one cannot be sure that any particular result will generalize to other plausible functional forms (many do not).

ii. The first 2/3s of the course uses explicit or concrete functional forms for most of the models explored.

- This is done because most students will find this easier, since concrete functional forms are used in most algebra and calculus classes.

- The last third will explore the extent to which similar results can be found using more general functions.

II. Supply and Demand as Implications of Unconstrained Net-Benefit Maximization

A. As mentioned in chapter 2, there are many settings in which one can model choices using results (or tools) from calculus that characterize unconstrained optimization.

i. A bit of work was done in chapter 2 to show how calculus can be used to characterize choice settings that can be analyzed using the calculus of unconstrained optimization. Generally, they involve objective functions that are strictly concave.

ii. We return to the-net-benefit maximizing model in this section of chapter 3 and use it as a basis for a theory of prices.

- In chapter two, the purpose of the net-benefit maximizing model was to illustrates how one can construct a mathematical model of individual choice.
- In this chapter its purpose is to use that model as to derive the demand for a good sold in a competitive market and use that model to characterize a market demand function when individuals in the market are very similar to one another.
- We then use the net benefit maximizing model to characterize the decisions of a profit maximizing firm that produces and sells goods in a competitive market.
- That model, in turn, can be used to characterize a market supply curve when suppliers are basically similar (have essentially the same cost functions).

iii. Together market supply and demand curve can be used to characterize equilibrium (market-clearing) prices and sales in that market.

B. The end of this chapter focuses on choice settings in which constraints must be taken into account. Both sets of results (tools) from calculus—both unconstrained and constrained optimization—are widely used in economics and in subsets of other social sciences that rely upon rational choice models.

- In the next chapter, we'll use constrained optimization to develop somewhat richer models of the behavior of consumers and firms in competitive markets.

C. A net-benefit maximizing model of individual demand.

i. Recall that a consumer's net benefits from purchasing Q units of the good can be written as:

$$N = b(Q) - c(Q)$$

ii. Suppose that a typical or average consumer's total benefits (in dollars) from using a good is $B = aQ - Q^2$, where Q is the quantity of the good consumed and a is greater than 0. Suppose also that the consumer can purchase the good for price P.

iii. In other words, suppose that we adopt an explicit functional form for benefits, $B= aQ - Q^2$ and for costs $C = PQ$. This makes $N = aQ-Q^2 - PQ$

iv. Differentiating N with respect to Q and setting the result equal to zero characterizes the net-benefit maximizing quantity of the good, which is the quantity that a net-benefit maximizer will purchase. $dN/dQ = a - 2Q - P = 0$

- (This equation is often referred to as the first order condition. If function N has a maximum at some value of Q, that value will necessarily satisfy the first order condition. It will be a value that makes the derivative of N equal to zero.)
- Note that the first terms (a-2Q) are his or her marginal benefit and the last (P) is his or her marginal cost.
- Note that we can solve the first order condition for Q as a function of P. We can add 2Q to both sides of the equation to make $a -P = 2Q$, dividing both sides by 2 and shifting the Q to the righthand side yields $Q^* = (a-P)/2$

v. Since the quantity that satisfies this equation (usually denoted as Q*) is the "optimal" one for this individual given **any particular price** confronted, the function that describes Q* is his or her demand function for that good.

- $Q^* = (a-P)/2$ describes how much this consumer buys for any "given" price.

vi. Solutions to the consumers net-benefit maximization problem for Q* thus characterize this consumer's individual demand function (or curve) for the good being purchased. This would be true of other functions as well.

- Note that for the demand function that we've found, as price increase demand falls, so this consumer's demand curve "slopes downward."

vii. We can use an individual's demand curve as the basis for characterizing market demand, if the person modelled is "typical" or "average." For example, if there are N such individuals in the market, overall demand at any particular price will be N times as great or

$$Q^D = NQ^*= N(a - P)/2$$

- (Note that **if consumers are not identical** or very similar, characterizing market demand requires adding up the demand functions for each individual consumer, rather than simply multiplying one demand curve by the number of consumers in the market.)

D. A very similar net-benefit maximizing approach can be used to model the supply side of the market.

i. Suppose that cost of production for a typical firm in the market is $C = cQ^2$, where c is greater than 0. Suppose also that the firm can sell as much as it wants to at prevailing market price P. Its net benefits are net revenues or profit (Π). This can be represented as total revenue less total cost, $\Pi = PQ - cQ^2$.

- (Note that profit is also a net benefit, where the firm owner's total benefit (revenue) from selling Q units of the good is PQ and his or her total cost is cQ^2.)

ii. The optimal quantity to produce for sale is that which maximizes profit. We can characterize that quantity by differentiating the profit function with respect to Q and setting the derivative equal to zero, which yields: $d\Pi/dQ = P - 2cQ = 0$

iii. To characterize this firm's quantity supplied for any particular price, we need only solve the first order condition, $P - 2cQ = 0$, for Q.

- The algebraic steps for doing so are similar to that used above to characterize demand above:
- Shift all the Q terms to the lefthand side of the equation and the others to the right hand side: $2cQ = P$.
- Divide both sides of the equation by 2c, which yields: $Q^* = P/2c$
- This is the supply function for the good of interest.
- Note that quantity produced and sold by this firm rises with the prevailing market price.

iv. If there are M firms in the market with similar cost functions, then the market supply function (or curve) is simply M times that of the typical or average firm, which is

- $Q^S = MQ^* = M\,P/2c$
- (Note that if firms are not identical or very similar, market supply requires adding up the supply functions of each firm, rather than simply multiplying one of the supply curves by the number of firms in the market. The assumption that suppliers have similar cost functions is sometimes called the Marshallian assumption about competitive markets.)

E. At the market equilibrium, price adjusts to set the quantity demanded [$Q^D = N(a - P)/2$] equal to the quantity supplied [$Q^S = M\,P/2c$].

i. The price that does so is usually denoted as P* and it sets $Q^D = Q^S$ or using our results from above, $N(a - P)/2 = M\,P/2c$

- To find the equilibrium price, shift all the P terms to the lefthand side and the others to the right which yields: $-NP - MP/2c = -Na$
- Multiply both side by -1 and factor the lefthand side: $P(N+M/2c) = Na$
- Multiply both sides by $1/(N+M/2c)$ which yields: $P^* = Na/(N+M/2c)$

- Which can also be written as P* = 2Nac /(2cN +M)

ii. One can find the total quantity sold in equilibrium by substituting P* into either the demand curve or supply curve if one has made no math errors in determining P*.

- (Think about this and explain why it doesn't matter which function you use.)
- I often use the supply curve, because it often has a simpler algebraic expression.

iii. One can also undertake comparative statics on market price and quantity by taking derivatives of P* or Q* with respect to the terms of interest—or simply look at the result and use your mathematical intuition to see what the effects would be.

- For example, if the number of firms increases, it is clear that the equilibrium price in this market would fall. (Note that M appears only in the denominator, thus as M rises the term describing the equilibrium market price falls.)

F. The particular form of the demand and supply curves derived, and the equilibrium prices will vary with the assumptions made about individual total benefit functions and firm total cost functions, but the basic logic and steps taken will be similar for all cases in which explicit functional forms are assumed for the consumer total benefit and firm total cost functions.

- Towards the end of the course, we will show how to characterize market supply and demand curves and market equilibria while making only assumptions about the general shapes of those functions.

i. As an exercise, change the explicit forms assumed and derive demand and supply curves and market equilibrium. For example, let a firm's cost function be $C = cQ^{.5}$ and/or let a typical consumer's total benefit function be $bQ^{.5}$

- (Hint: such functional forms generally require taking somewhat odd "roots" to characterize their associated demand and supply curves. Note that a consumers demand curve is determined by his or her total benefit function and that a firm's supply curve is determined it total cost function.)

III. Models of Market Behavior Using the Calculus of Constrained Optimization

A. Although the calculus of unconstrained optimization can shed a good deal of light on decisions made by firms and consumers, there are many cases in which taking account of particular constraints can shed additional light on such decisions.

- For example, taking account of a consumer's budget constraint allows one to characterized tradeoffs in their decisions about how to allocate their budget among the goods that they might purchase.

B. If constraints are important, then other methods for finding the "optimum" for consumers and firms are necessary. Two of the most widely used results (tools) from calculus are the "substitution method" and the "Lagrangian method."

- The **substitution method** can be used when the constraints can be incorporated into the objective function in a manner that creates an unconstrained optimization problem. (That method tends to focus the optimization along the edge(s) of the constraint(s) of interest, and often yields relatively straightforward first order conditions with easily understood economic implications.)

- The **Lagrangian method** involves "adding" the constraint (in a particular form) to the objective function in a manner that causes the derivatives of the new function to take account of the constraint.

- We will spend less time on the Lagrangian methods in this course than most math-econ textbooks do because it is rarely used in the economic literature these days. (It was very commonly used in the 1960s and 1970s.) However, it turns out to be quite useful for working with multiplicative exponential functions, so we'll be using it quite a bit at first.

C. **The Substitution Method**. In many cases, it is possible to "substitute" the constraint(s) into the objective function (the function being maximized) to create a new composite function that fully incorporates the effect of the constraint.

i. For example, consider the separable utility function: $U = x^{.5} + y^{.5}$ to be maximized subject to the budget constraint $100 = 10x + 5y$. (The consumer has 100 dollars to spend. Good x costs 10 $/unit and good y costs 5 $/unit.)

ii. Notice that the constraint implies that we can write y as: $y = [100 - 10x]/5 = 20 - 2x$

- Substituting that characterization of y into the objective function for y yields a new function now written entirely in terms of x: $U = x^{.5} + (20 - 2x)^{.5}$
  - This function accounts for the fact that every time one purchases a unit of x one has to reduce his or her consumption of y.
  - So, the new function includes the entire effect of the constraint on purchases of x.
- Differentiating with respect to x allows the utility maximizing quantity of x to be characterized as:
  - $d[x^{.5} + (20 - 2x)^{.5}]/dx = .5x^{-.5} + .5(20-2x)^{-.5}(-2)$
  - This derivative has the value zero at the constrained utility maximum.
  - Note that the first term is the marginal benefit in utility terms and the second term is the marginal opportunity cost of consuming X in utility terms (from reducing purchases of Y).
- Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for x yields:
  - $4x = 20 - 2x$, which can be simplified to $6x = 20$, which implies that $x^* = 3.33$

- Substituting x* back into the budget constraint yields a value for y*
  y = 20 - 2(3.33) which implies that y* = 13.33

- No other point on the budget constraint can generate higher utility for this consumer than that at (x*,y*) =(3.33,13.33).

D. **The Lagrangian Method.** The Lagrangian method is less intuitive than the substitution method. It adds the constraint(s) (such as the budget constraint) to the objective function (such as a utility function) in a specific form. First, it alters the budget constraint so that it equal's zero, such as $0 = W - P_1X_1 - P_2X_2$. Then it multiplies that function by a term normally called the Lagrangian multiplier, which is usually denoted with the Greek letter lambda (λ). The resulting function is called the Lagrange function.

i. To illustrate how to use the Lagrange method, we'll apply it to the consumer choice problem worked out under "C." Recall that the objective function was $U = x^{.5} + y^{.5}$ and the budget constraint was 100 = 10x + 5y.

- The Lagrange function is: $L = x^{.5} + y^{.5} + \lambda(100\text{-}10x - 5y)$.
  - (The objective function is utility, $x^{.5} + y^{.5}$, and the constraint has been put a form so that it equals zero and then multiplied by lambda.
- To optimize the Lagrange function, one takes (partial) derivatives with respect to all the control variables (in this case, the variables that the individual consumer controls, x and y) plus a derivative for lambda and set each derivative equal to zero.
- Note that **partial derivatives** are simply derivatives of the function of interest (here the one characterizing L) that assume that all the other control variables are constants (do not change) while taking account of the changes induced by the variable being differentiated. This process is repeated for each control variable—and also for each Lagrangian multiplier.

ii. The result is a system of equations referred to as the "first order" conditions.

- $dL/dx = .5x^{(.5-1)} - 10\lambda = 0$
- $dL/dy = .5y^{(.5-1)} - 5\lambda = 0$
- $dL/d\lambda = 100\text{-}10x\text{-}5y = 0$

iii. All three of these conditions hold simultaneously, so the algebra used to solve for the utility maximizing level of x and y is a bit more complicated than in the previous illustrations. One of the usual methods for solving for x and y involves eliminating the lambdas from the equations.

- Add the negative of the lambda terms to both sides of the two equations that include lamda(s) to make
  - $.5x^{(.5-1)} = 10\lambda$

- $.5y^{(.5-1)} = 5\lambda$

- Divide the top equation by the second to obtain:
  - $(x/y)^{(.5-1)} = 2$
  - Which the exponent implies can be simplified to $(y/x)^{(.5)} = 2$
  - Squaring both sides yields $(y/x) = 4$
  - Which implies that $y = 4x$
- Substitute this expression for y into the last equation (the constraint) which yields
  - $100-10x-5(4x) = 100-30x = 0$
  - Solving for x implies that $x^* = 100/30 = 3.33$,
  - which given that $y = 4x$ implies that $y^* = 13.33$
- Note that this result is exactly the same as the one found using the substitution method.
- In general, economists use which ever method is easiest (or possible) to apply for the constrained optimization problem being worked on.

E. In this class, we'll be mostly using whichever of the Substitution and Lagrangian methods is easiest in most cases—except initially when students are trying to understand both methods.

- The Lagrange method is somewhat more flexible and general than the substitution method, but it is usually more difficult to generate clear solutions with it and somewhat more difficult to understand.

IV. Some Practice Problems

A. Use the substitution method to:

i. find the utility maximizing level of goods g and h in the case where

$$U = g^a h^b \text{ and } 10 = g + h,$$

ii. Find the utility maximizing bundle of goods when the budget constraint is $20 = g + h$ (i.e. if the wealth constraint is twice as high).

iii. Repeat this problem using the Lagrangian method.

B. Characterize the profit maximizing output of a firm when $\Pi = PQ - aQ - Q^2$.

C. Consider the demand function $Q = a + bP + cY$, with $b < 0$ and $c > 0$.

- Find the slope of this demand function in the QxP plane.
- Find the slope of this demand function in the QxY plane.
- Is the associated revenue function (R=PQ) concave? strictly concave?

- (Hint, use the demand function to restate price as a function of quantity and then take the first and second derivatives of the revenue function with respect to Q.)
- Characterize the profit maximizing output of this firm and discuss briefly the meaning of the various terms in your equation(s).

V. Appendix: A Few More Useful Ideas and Definitions from Calculus

A. Some useful relationships, concepts, and definitions from and for Calculus

i. Given Y = abcdX , the first derivative of Y with respect to X is dY/dX = abcd

ii. Given Y = aX^b , the first derivative of Y with respect to X is dY/dX = abX^{b-1}

- o When b-1=-c < 0 then abX^{b-1} can be written as ab/X^c
- o This is sometimes useful when using algebra to isolate all the X terms on one side of an equal's sign.

B. Def: Function Y = f(X) is said to be **continuous** whenever the limit of f(X) approaches Y= f(Z) as X approaches Z.

- Or alternatively, function Y = f(X) is said to be continuous if for every point in the domain of X, and for any e >0, there exists d > 0, such that |f(X) - f(Z)|< e for all X satisfying |Z - X| < d.
  - o That is to say, points only a finite distance from Z should generate function values within a finite distance of f(Z).
  - o In fact, function f is continuous if for any finite distance e (epsilon) there exists d (delta) such that any value within delta of z generates a function value within epsilon of f(z).)
  - o This condition assures that there are no "sudden" (instantaneous) jumps in the function and no holes in the function.
  - o [Note that Y = 1/X is not continuous at 0, because as one gets close to Zero Y increases by more and more, some of these increases will exceed the "d" chosen.]
- Def: the limit of a function: function f is said to have a **limit point** y* at x* if and only if (iff) for every e > 0, there is a d > 0 such that |f(x) - y*| < e whenever |x - x*|<d.
  - o If there is a real number y* satisfying this definition at x*, we the limit of f at x* exists and equals y*.
  - o Note that this definition rules out the existence of different right hand and left hand limits. (why?)
- Def: function f is said to be **differentiable** if and only if (iff) for every x contained in set X the limit point of { [(f(x) - f(z)]/(x - z) } exists.

- Note that if f is differentiable, f is also continuous. (why?)

C. Within microeconomics, utility functions and production functions are generally assumed to be continuous and twice differentiable.

i. Such assumptions clearly rule out some kinds of *decision makers,* just as the assumption that production possibility sets and opportunity sets are convex and compact rule out some kinds of *choice settings*.

ii. These assumptions are made largely for "economic" rather than "empirical" reasons.

- That is to say, generally it is felt that the benefits of more tractable models overwhelm the costs of reduced realism and narrower applicability.

iii. However, this assumption should not always be taken for granted. There are a few cases in which continuous versions of the choice settings lead to empirically false predictions.

- Clearly, such uncommon choice settings should not be entirely neglected.
- When discrete aspects of the choice problem are, or may be, important, various tools from set theory, integer programming, geometry, and real analysis can still be applied.

iv. However, for most choice settings of interest to economists, the assumption of continuity is approximately correct. There may be a smallest piece of flour, sugar, gasoline, or sand, but they are pretty small!

(Try to think of cases where the simplifying assumptions of continuity and convexity will generate predictions about behavior that are clearly wrong.)