I. Rationality and Constrained Optimization

A. This chapter shows how the mathematics of constrained optimization can be used to model economic choice settings similar to those of chapter 3, but with more aspects of consumer and producer choices taken into account.

B. Applications of the calculus of constrained optimization show how economic models can be improved by including a richer model of rational choices, as they have been from the late nineteenth century onwards—although the core models developed in the previous chapter have not changed very much in the past 80 years.

C. The constrained optimization conception of rationality allows one to use calculus to model the choices made in circumstances in which there are obvious constraints—such as budget constraints or technological ones—that limit was is feasible for the decision maker of interest.

D. The process of building models of constrained optimization involves several step. (i) First, one has to characterize the goals of the decisionmaker (the objective function) and means available for realizing that objective (the decisionmaker's control variables). (ii) Then, one has to characterize what is possible (the feasible set). The feasible set is characterized by various constraints faced by individuals and organizations such as firms. Constraints include, for example, ones characterized by a consumer's budget, his or her knowledge, the available technology for making and buying things and services. Constraints also include laws of various kinds that limit what can be done with the resources possessed by individuals and organizations, including criminal laws, civil laws, tax laws, safety laws, and so on.

i. As mentioned in chapter 3, objectives are normally characterized as functions that map actions of various kinds into real numbers that are assumed to measure or be correlated with and individual's or organization's objectives (net benefits, profits, utility).

ii. The constraints on one's choices may or may not always be binding, but whenever they are binding (or likely to be binding), they effect choices through effects on the feasible domain of actions and outcomes. In this lecture, we'll only consider cases where the constraints are binding and representable with equations.

- Some constraints such as a budget constraint are inherently numerical. One only has so much money to spend or so large of a credit line, and prices are naturally real numbers.

- Other constraints may not be entirely measurable, such as the extent of one's knowledge or ignorance. But, if the terms "more" and "less" are meaningful (as with more or less ignorant), we can represent such constraints and the effects of such constraints (such as errors) using mappings into real numbers.

- These numerical representations usually characterize constraints as equations, which together with an individual's or group's objective function(s) allows the calculus of constrained optimization to be used to characterize decisions and many market outcomes.

II. Supply and Demand as Implications of Constrained Optimization

A. Chapter 3 developed the basic tools of constrained optimization from that calculus that we'll be using in this chapter: the substitution method and the Lagrangian method.

i. In chapter 3, it was demonstrated how those methods could be used to characterize how a utility maximizing individual's allocation of his or her budget.

ii. In this section of chapter 4, we'll use a slightly more abstract version of that model of consumer choice to provide a richer basis for a theory of individual and market demand.

iii. A somewhat similar approach can be used to characterize the cost functions of firms, which can be used to provide a richer theory of firm decisions and market supply.

iv. Together these results can be used to provide a more complete model of the factors that determine prices in competitive markets.

- Chapter 5 will show how these same results can be used to model the pricing and output decisions of firms in monopoly and other less than full competitive market settings.

B. **A constrained utility maximizing model of individual demand.**

i. Suppose that a typical or average consumer is allocating his or her budget between two goods, X_1 and X_2. Suppose that he or she has W dollars to spend and that the price of good X_1 is P_1 and the price of good X_2 is P_2. Assume that the consumer's utility function is $U = X_1^a X_2^b$ with $0<a<1$ and $0<b<1$.

ii. This is a constrained optimization problem. Our consumer, who we'll refer to as Al (short for Allan or Alice), wants to find the combination of X_1 and X_2 that maximizes his or her utility given her budget constraint, $W = P_1X_1+P_2X_2$.

iii. Either the substitution or Lagrangian method can be used, but given an exponential utility function, the Lagrange method is often easier to apply, and so, we'll use that method. (Such functional forms are rare cases in which the Lagrange method involves less complicated calculus and algebra than the substitution method.)

iv. The first step is to form the Lagrangian equation. Recall that the Lagrangian function is formed by adding a constraint to the objective function (here the utility function) in a particular form. First one puts the constraint into a form that equals zero, and then multiplies that function by lambda. Next one adds this to the objective constraint generates the Lagrange function: $L = X_1^a X_2^b + \lambda(W - P_1X_1 - P_2X_2)$.

- Note that the first term is the objective function (the utility function) and the second is the constraint in a form equal to zero (from the budget constraint) multiplied by the Lagrange multiplier.

v. There are **two control variables** (the "things" that the chooser can change to increase or decrease values of their objective function), here purchases of X_1 and X_2, and **one Lagrangian multiplier** (λ), so there are **three partial derivatives** that need to be characterized and set equal to zero.

- (The Lagrangian method can include more than one constraint, although this is not common in economic models. If there are several constraints, there are different Lagrange multipliers for each constraint.)

vi. The three first order condition (first derivatives set equal to zero) are:

- $dL/dX_1 = aX_1^{a-1} X_2^b - \lambda P_1 = 0$
- $dL/dX_2 = bX_1^a X_2^{b-1} - \lambda P_2 = 0$
- $dL/d\lambda = W - P_1X_1 - P_2X_2 = 0$
- (Note that **partial derivatives** are simply ordinary derivatives that assume that all the other control variables and parameters of the problem (such as exponents, wealth and prices) are constants.)
  - The main calculus formula to keep in mind here are if $Y = aX^b + c$ then $dY/dX = abX^{b-1}$ (Notice that the "c" disappears and the "a" is not affected by the derivative.)
  - When taking partial derivatives only the terms (or functions) with the relevant variables are of interest, the rest—including other terms that you'll be taking derivatives of—are of the a, b, c variety. Thus, for the partial derivative with respect to X_1, the X_2 terms are all analogous to the terms "a" or "c" in this rule from the calculus of one variable.

vii. The solution is found in a manner similar to that used in the consumer allocation model developed towards the end of chapter 3—although this time, rather than a numerical solution, **two demand functions** are characterized.

viii. Here is one set of algebraic steps that can be used to characterize those demand curves.

- First shift the lambda terms in the first two "first order conditions" to the righthand side of the equal sign and then divide the first equation by the second. (One can also divide the second by the first if you wanted to.)
- The first step yields: $aX_1^{a-1} X_2^b = \lambda P_1$ and $bX_1^a X_2^{b-1} = \lambda P_2$
- Second, divide the first equation by the second to get:

$$(aX_1^{a-1} X_2^b / bX_1^a X_2^{b-1}) = \lambda P_1 / \lambda P_2$$

- Which simplifies to $(aX_2 / bX_1) = P_1 / P_2$
  - Recall from algebra that $X^{-b} = 1/X^b$, that $X^b/X^c = X^{b-c}$, that $X^bX^c = X^{b+c}$, and that $[aX^b]^c = a^cX^{bc}$
  - All four of these algebraic relationships are very useful in working through these problems.

ix. Suppose that we are interested in Al's demand for X_1. In that case, we want to solve for Al's ideal value of X_1 as a function of prices, wealth, and other terms.

- To do so we take the last line of the above and solve for X_2 as a function of X_1 and the other terms.
- We then substitute that expression for X_2 into the budget constraint for X_2 (the third first order condition).
- Multiplying both sides by bX_1 and then dividing each by "a" yields:
  $X_2 = (bP_1 / aP_2)X_1$
- Substituting this for X_2 in the budget constraint yields:
  $W - P_1X_1 - P_2[(bP_1 / aP_2)X_1] = 0$
- Next, we solve for X_1. This produces a function that characterizes Al's ideal choice of X_1, which we refer to as X_1^*, as a function of prices and wealth. X_1^* is the quantity of X_1 that maximizes Al's utility given his or her utility function given any prices for the two goods and his or her wealth.

Steps that allow you to isolate X_1 are listed below. (You may use others; as long as the algebra is correct, you'll generate the same or an equivalent result.)

- $W - P_1X_1 - [(b/a) P_1X_1] = 0$
- $X_1 (P_1 + (b/a)P_1] = W$
- $X_1 P_1 (1 + b/a) = W$
- $X_1^* = W / [P_1 (1 + b/a)] = W / [P_1 (a/a + b/a)]$
- $X_1^* = W / [P_1 ([a + b]/a)] = [a/(a+b)] W/P_1$
- **The last line is Al's demand function for good X_1:** $X_1^* = [a/(a+b)] W/P_1$

- Note that the amount that Al spends on X_1 is P_1X_1* which is a **constant fraction** of his or her budget, [a/(a+b)] W , in this case.
  - How many units purchased depends on the price.
  - The quantity purchased is, in this case, the ideal fraction of one's budget constraint [a/(a+b)] for purchases of X_1 times one's budget (W) divided by the current price of the good (P_1).
  - This is one of the **odd properties** of exponential utility functions, although it tracks the demand for some goods fairly well in statistical studies.
  - **The demand for the other good, X_2, is very similar**:

$$X_2^* = [b/(a+b)] W/P_2$$

- Note that the demand for either good varies with the exponents of the utility function (tastes) and one's budget (W) which may be thought of as one's wealth or income.
  - As income rises, expenditures rise proportionately to the increase in income. A doubling of income, doubles expenditures.
  - All goods are normal goods if consumers have multiplicative exponential utility functions with positive exponents.
- Note also that Al's demand curve slopes downward in the price domain, as P_1 increases, the quantity purchased falls.
  - All demand curves slope downward if consumers have multiplicative exponential utility functions with positive exponents.

x. If Al is the average individual or there are N such individuals in the market, the overall demand at any particular price will be N times as great as Al's, or:

$$X_1^D = N\,[a/(a+b)]\,W/P_1$$

- As in the simpler model of market demand developed in Chapter 3, **if consumers are not identical** or very similar, characterizing market demand requires adding up the demand functions for each individual consumer, rather than simply multiplying one demand curve by the number of consumers in the market.
- In such more complicated cases, $X_1^D = \Sigma X_{1j}^*$ with j = 1, 2 … N
- In this class, we'll mostly assume that we are modeling the "**average consumer**" and so simply maximize our result by the number of consumers, but this is not always the best approach. There may, for example, be several distinct but different types of consumers.

C. A very similar approach can be used to model the supply side of the market. Unfortunately, the results are more somewhat more difficult to derive and more complex. Nonetheless, the results cast useful light on the factors that influence market supply.

i. Suppose that cost of production for a typical firm in the market is C = wL + rK where w is the cost of labor (L) and r is the cost of capital (K).

ii. As we'll see, the costs depend on factor prices (w and r) and the production function used to produce the products to be sold. Suppose also that the firm's maximal output from the use of labor and capital is $Q = L^eK^f$, where both e and f are greater than 0, but less than 1.

- Note that the "technology" of production, in principle, affects the functional form of the production function, which in this period is: $Q = L^eK^f$.

    - **Technology** (including both management and engineering) determines both the exponents and the fact that the production function has the multiplicative exponential form in this case.

- That the exponents are less than 1 and greater than zero implies that the each factor of production exhibits diminishing marginal returns. If their sum is less than 1, this implies that the overall production process does as well.

    - If their sum is exactly 1, as in **Cobb-Douglas** production functions, then the overall production process exhibits constant returns to scale.

iii. The firm's cost function can be derived either by minimizing the cost of a given output or by maximizing the output achieved for a given cost. Each of these approaches can be regarded as the "**dual**" of the other.

- We'll first use the maximizing output for a given cost approach, because this is similar to that used for deriving consumer demand and is, in some sense, the "natural" way to characterized a firm's cost function. Unfortunately, the results are more difficult to derive and interpret than the solution to the dual problem—as we will see.

    - The solutions obtained are similar to consumer problem above, in that they characterized a firm's demand for inputs for a given overall expenditure on inputs, although the objective function that we are modelling at this point output or production (Q), rather than utility.

- We can characterize the firms demand for inputs by using the production function as an objective function and the cost function as a constraint. Again, the Lagrange approach is a bit easier, because the objective function is a multiplicative-exponential function. (But, as usual, the individual equations are more difficult to interpret than using the substitution method.) Since L is begin used to characterize the quantity of labor used in production, I use a "script L" as the name of the Lagrange function ($\mathscr{L}$).

- The Lagrangian equation is $\mathscr{L} = L^e K^f + \lambda(C - wL + rK)$
- As in the consumer choice model, there are 3 first order conditions, two with respect to the control variables (L and K) and one with respect to the Lagrangian multiplier. (There would be more first order conditions if there were more control variables as with 2 kinds of labor or more constraints.)

  ○ $d\mathscr{L}/dL = eL^{e-1}K^f - \lambda(w) = 0$

  $d\mathscr{L}/dL = fL^eK^{f-1} - \lambda(r) = 0$

  $d\mathscr{L}/d\lambda = (C - wL + rK) = 0$

- To find the input demands of the firm, we follow the same steps as in the previous consumer constrained optimization problems: shift the lambda terms to the righthand side of the equations and divide one equation by the other to generate:
  - $eL^{e-1}K^f/fL^eK^{f-1} = w/r$
  - Which simplifies to: $eK/fL = w/r$
- If we want the firm's demand for labor we solve this equation for K and then substitute that result into the constraint (derivative of the lambda term) ○ $K = (w/r)(f/e) L$
- so $C = wL + r (w/r) (f/e) L$
- Reversing the sides and factoring yields:

  $L (w + w(f/e)) = Lw(1+f/e) = C$

  So, $L^* = (C/w)(1/(1+f/e)) = [e/(f+e)] [C/w]$

- Notice that this expression looks just like the expression that we found for the consumer demand function, but in this case, it characterizes this firm's demand for labor for a given expenditure on inputs (C).
- A similar result can be obtained for the firms demand for capital.
  $K^* = [f/(f+e)] [C/r]$

- Notice also that the pattern of input demand is determined by the relative productivity of the inputs (as reflected in their respective exponents), the amount that the firm plans to spend on all of its inputs (C), and input prices (the wage rate (w) and the cost of capital (r)).

iv. However, the cost function we need describes costs in terms of outputs. What we have at this point is the ability to describe outputs in terms of expenditures on inputs.

- If we know how much money is spent on all inputs, we also know how many of each of the inputs are employed. This allows us to determine how much output is produced using the production function.

- This can be determined by substituting the ideal input quantities into the production function.

    - Recall that the firm's output is $Q = L^e K^f$

- Our two input demand functions allow the firms output to written in terms of production costs as:

  $$Q = \{[e/(f+e)]\ [C/w]\}^e\ \{[f/(f+e)]\ [C/r]\}^f$$

- Note that this characterizes output in terms of overall expenditures on inputs, their productivity (as characterized by the exponents) and input prices.

- Note also that we can solve for C (the cost or expenditures on inputs) by factoring it out of the righthand side expression fairly easily

    - $Q = C^{e+f}\ \{[e/(f+e)]\ [1/w]\}^e\ \{[f/(f+e)]\ [1/r]\}^f$

- This allows Cost (C) to be written as a function of output (Q), which is what we need for a total cost of production function.

  $$C^{e+f} = Q\ /\{[e/(f+e)]\ [1/w]\}^e\ \{[f/(f+e)]\ [1/r]\}^f$$

  $$C^* = \{Q\ /\{[e/(f+e)]\ [1/w]\}^e\ \{[f/(f+e)]\ [1/r]\}^f\ \}^{1/(e+f)}$$

  The above characterization of C* is the firm's cost function.

v. **Alternatively, we can use the "dual" of the firm's optimization problem to get at the cost function instead.**

- **In some cases, this yields a cleaner and more direct result.**
- The dual choice problem requires us to minimize cost (expenditures on inputs) subject to producing a given output Q.

vi. Essentially, the dual just reverses the objective function and constraint.

- The new Lagrangian function is:
- $\mathscr{L} = - wL + rK + \lambda(Q - L^eK^f)$
- Again there are 3 first order conditions, two with respect to the control variables (L and K), and one with respect to the Lagrangian multiplier. The first two are very similar to those we derived before, but the last is quite different.

  o $d\mathscr{L}/dL = w - \lambda(eL^{e-1}K^f) = 0$ o

  $d\mathscr{L}/dL = r - \lambda(fL^eK^{f-1}) = 0$ o

  $d\mathscr{L}/d\lambda = (Q - L^eK^f) = 0$

- (I've again used a script L ($\mathscr{L}$) for the Lagrangian equation, because L is being used for the quantity of labor employed producing the good of interest.) Notice that the only significant difference in the first order conditions is the derivative with respect to the Lagrangian multiplier, λ.
- The same steps are undertaken as before, but in this case solutions will be in terms of output (Q) rather than expenditures on inputs (C).
  - Shifting the lambda term to the right and dividing yields: $w/r = eK/fL$
  - If we again focus on labor initially, we want to again specify capital in terms of labor, which is again $K^* = (fw/er) L$

  Substituting this into the production function and solving for L, again takes a few steps:

  - $Q = L^eK^f = L^e[(fw/er) L]^f$ o L can be factored out of the righthand expression

    $Q = L^{e+f}(fw/er)^f$

  - We can then solve for L* in terms of Q: $L^* = [Q (er/fw)^f]^{1/e+f}$
  - Recall that $(x/y)^{-e} = (y/x)^e$, thus the ratio inside the brackets "flips" as one derives L*
  - This characterizes the **demand for labor as a function of output**, productivity (again indicated by the exponents) and the price of labor and capital (w and r).

- We can solve for K* in a similar way.
  - Isolating the L (instead of K) yields: $w/r = eK/fL$ yields $L = (e/f)K (r/w)$

- Substituting this into the constraint yields $Q = L^e K^f$ $= [(e/f)K\,(r/w)]^e K^f$ o The K can be factored out $Q = K^{f+e}\,[(er/fw)]^e$
- Solving for K yields $K^* = [Q\,(fw/er)^e\,]^{1/f+e}$
- This characterization of K* is the **firm's demand for capital as a function of output**, input prices, and their productivities.

• The cost function can now be written in terms of the optimal quantity of labor and capital for various quantities of output:

- $C = wL^* + rK^* = w\,[Q\,(er/fw)^f\,]^{1/e+f} + r\,[Q\,(fw/er)^e\,]^{1/f+e}$
- Note that the first term is the firm's expenditure on labor and the second is the firm's expenditure on capital used in production in the **optimal amounts** for the output quantity of interest.
- Note also that it is a simpler expression than the one derived the first way, although they should be mathematically equivalent as long as we've made no algebraic errors.

• **This is a much simpler**, if somewhat less intuitive solution than the one solved directly.

- Notice in both cases, production costs vary with technology (the size of the exponents) and input prices.
- Costs clearly rise with input prices (recall that the exponents are less than 1) and costs tend to fall as the sum of the exponents fall.
- (Note that both the first and second derivations of the firms total cost function have characterized **long run total costs**, because the firm has been assumed to be able to vary all of its inputs.)

We'll use this simpler expression to derive the firm's supply curve.

• To further **simply the notation,** let's make $m^L = (er/fw)^f$, $m^K = (fw/er)^e$, and $\alpha = 1/(f+e)$.

- This lets us write the cost function directly above as:

$$C = w\,(Qm^L)^{\alpha} + r\,(Qm^K)^{\alpha}$$

- The replacement notations are for clusters of variables that do not change when we calculate profit maximizing outputs—but would change if wages, capital rental costs or technology change. The simpler notation reduces the likelihood of algebraic mistakes in deriving the supply curve. After that is complete we can (and probably should) substitute the "real" expressions behind the three new terms back into the equation worked out.

- The firm's profit maximizing output is calculated in the same manner as in chapter 3.
  - Profit is total revenue (PQ) less total cost (now written as $C = w\,(Qm^{L})^{\alpha} + r\,(Qm^{K})^{\alpha}$).
  - $\Pi = PQ - w\,(Qm^{L})^{\alpha} - r\,(Qm^{K})^{\alpha} = PQ - wQ^{\alpha}(m^{L})^{\alpha} - rQ^{\alpha}(m^{K})^{\alpha}$ o

    Differentiating with respect to Q yields:
  - $P - \alpha\, wQ^{\alpha-1}(m^{L})^{\alpha} - \alpha\, rQ^{\alpha-1}(m^{K})^{\alpha} = 0$ at Q*
  - The first term (P) is marginal revenue, the others are the firm's marginal cost. The individual terms show the part of marginal cost attributable to labor costs and to capital costs.
  - (Note that **we have derived long run total cost** rather than short run marginal cost, because we are assuming that both labor and capital can be varied in the period of interest. So, this first order condition characterizes the firm's long run profit maximizing output.)
  - (Short run cost and supply would be derived by holding the quantity of capital or some other input(s) constant, which would be quite a bit simpler in the two-input production case.)
- One can solve for Q* (the profit maximizing output) by shifting P to the righthand side, multiply both by negative 1 and factoring.
- $\alpha\, wQ_{\alpha-1}(m_{L})_{\alpha} + \alpha\, rQ_{\alpha-1}(m_{K})_{\alpha} = P$
- $Q^{\alpha-1}[\alpha\, w\,(m^{L})^{\alpha} + \alpha\, r\,(m^{K})^{\alpha}] = P$
- $Q^{\alpha-1} = P / [\alpha\, w\,(m^{L})^{\alpha} + \alpha\, r\,(m^{K})^{\alpha}]$
- $Q^* = \{ P / [\alpha\, w\,(m^{L})^{\alpha} + \alpha\, r\,(m^{K})^{\alpha}]\}^{1/(\alpha-1)}$
- Notice that this **firm's long run supply curve** is upward sloping in price and the quantity supplied at every price tends to fall as input prices rise (e.g. if either w or r increase—although fully determining this requires checking the derivatives of m^{L} and m^{K} to know for sure).

D. If there are M firms in the market with similar cost functions, then the market supply function (or curve) is simply M times that of the typical or average firm, which is

- $Q^S = MQ^* = M \{ P / [\alpha w (m^L)^{\alpha} + \alpha r (m^K)^{\alpha}]\}^{1/(\alpha-1)}$

  (As before, if firms are not identical or very similar, market supply requires adding up the supply functions of each firm, rather than simply multiplying one of the supply curves by the number of firms in the market. The assumption that suppliers have similar cost functions is sometimes called the Marshallian assumption about competitive markets.)

E. At a competitive equilibrium for good 1 above (X_1), price adjusts to set the quantity demanded of that good equal to its supply at that price.

vi. The algebraic method used to find the **market clearing price** is similar to that which we undertook with the simpler net-benefit maximizing model. We again want to find the price that sets demand equal to supply.

i. Demand is: $X_1^D = N [a/(a+b)] W/P_1$

ii. Supply is: $Q^S = M \{ P / [\alpha w (m^L)^{\alpha} + \alpha r (m^K)^{\alpha}]\}^{1/(\alpha-1)}$

iii. To find the equilibrium price is again a matter of algebra. We want to find P such that

- $N [a/(a+b)] W/P_1 = M \{ P / [\alpha w (m^L)^{\alpha} + \alpha r (m^K)^{\alpha}]\}^{1/(\alpha-1)}$
- To solve for the market clearing price, first factor out the P on the right
- $N [a/(a+b)] W/P_1 = M P^{1/(\alpha-1)} \{ 1/[\alpha w (m^L)^{\alpha} + \alpha r (m^K)^{\alpha}]\}^{1/(\alpha-1)}$
- Then multiply both sides by P
- $N [a/(a+b)] W = M P^{\alpha/(\alpha-1)} \{ 1/[\alpha w (m^L)^{\alpha} + \alpha r (m^K)^{\alpha}]\}^{1/(\alpha-1)}$
- Shift the P to left and divide to isolate P
- $P_{\alpha/(\alpha-1)} = \{N [a/(a+b)] W\} / \{ M P_{\alpha/(\alpha-1)} \{ 1/[\alpha w (m_L)_{\alpha} + \alpha r (m_K)_{\alpha}]\}_{1/(\alpha-1)} \}$
- Raise both sides to the $(\alpha-1)/\alpha$ power.
- $P^* = [\{N [a/(a+b)] W\} / \{ M P^{\alpha/(\alpha-1)} \{ 1/[\alpha w (m^L)^{\alpha} + \alpha r (m^K)^{\alpha}]\}^{1/(\alpha-1)} \}]^{(\alpha-1)/\alpha}$

i. This is the **equilibrium price** or market clearing price of good X_1.

ii. One can find the **total quantity sold** in equilibrium (Q^*) by substituting P^* into either the demand curve or supply curve if one has made no math errors in determining P^*. (Think about this, and explain why it doesn't matter which function you use.)

iii. One can also undertake **comparative statics** on market price and quantity by taking derivatives of P* or Q* with respect to the terms of interest—such as consumer income or the cost of capital—or simply look at the result and use your mathematical intuition to see what the effects would be (as we did with the cost function above).

- For example, if the number of firms increases, it is clear that the equilibrium price in this market would fall. (Note that M appears only in the denominator, thus as M rises the term describing the equilibrium market price falls.)

  iv. The particular form of the demand and supply curves derived, and the equilibrium prices will vary with the assumptions made about individual utility functions and firm's production function, but the basic logic and steps taken will be similar for all cases in which explicit functional forms are assumed for the consumer utility and firm production functions.

- Towards the end of the course, we will show how to characterize market supply and demand curves and market equilibria, while making only assumptions about the general shapes of those functions. (This, perhaps surprisingly, is much simpler although the comparative statics tend to be more complex.)

- As an exercise, change the explicit forms assumed and derive demand and supply curves and market equilibrium. For example, let a firm's production function be $Q = L^{.8} + K^{.9}$ and/or let a typical consumer's utility function be $U = aX_1^{.7} + bX_2^{.9}$

III. Some Useful Economic Implications of the Mathematical Results of this Chapter

A. The point of the math in this chapter is to show how to develop more sophisticated models of demand and supply than possible with demand and supply diagrams (where one has to intuit the factors that cause demand and supply curves to shift, rather than derive them) and with simpler net-benefit maximizing models (that tend to lack natural places for relative prices and income in models of demand and/or that lack obvious places for input prices and technology in models of supply).

- The above models have natural places for all these variables and also has plausible implications about their effects on market prices and outputs, effects that in principle can be tested empirically.

- The results do not merely affirm some of our intuitions. It shows how the various factors (parameters of consumer and firm choices) interact to produce a market equilibrium.

- In a principles of micro-economics course, shifts and supply and demand emerge because of claims made by one's instructor or textbook.

- In this course, they shift because of the model developed, and the reason for such shifts is provided by the logic of mathematics rather than intuitive story telling.

B. An additional use of such models is that they can be used to flesh out simpler models, as for example, by adding input prices directly into cost functions or income into demand functions based on net-benefit maximization.

- That is to say, after you understand how the parts of a model fit together in a concrete functional form (here the multiplicative exponential form), one can incorporate them into both simpler and more abstract models in the proper way to capture some of their effects on individual decisions and market outcomes.

C. In general, for your development as economists, it is more important to understand the logic behind the mathematics than to be able to exactly replicate it—although mathematical models will take up the largest part of the exams in this class. (This is after all a mathematical economics course.)

- However, you should also note that the derivation above for market equilibrium is far too long and complex to place on an exam.

- So, at most you will see pieces of it such as derive a firm's demand for an input or derive an individual's demand for a good (such as X1) using utility functions and a budget constraint.

IV. Some Practice Problems

A. Use the substitution method to:

i. Characterize the utility maximizing level of goods g and h in the case where

$$U = g^a h^b \text{ and } W = gP_g + hP_h$$

ii. Repeat this problem using the Lagrangian method.

B. Characterize the profit maximizing output of a firm when $\Pi = PQ - aQ + Q^2$.

C. Consider the demand function $Q^D = a + bP^2 + cY^2$, with $b < 0$ and $c > 0$.

- Find the slope of this demand function in the QxP plane.

- Find the slope of this demand function in the QxY plane.

- Is the associated revenue function (R=PQ) concave? strictly concave?

  - (Hint, use the demand function to restate price as a function of quantity and then take the first and second derivatives of the revenue function with respect to Q.)

D. Use the production function $Q = aL^{.5} + bK^{.5}$ to (1) characterize the marginal product of labor and capital, (2) to characterize the cost function of a firm facing a price of w for labor and r for capital, and (3) to characterize the firm's profit maximizing output.