EC 421 Chapter 10: Game Theory and Models of Duopoly

I. Where We Are.

A. During the first half of the semester, we used mathematical models of rational decision making (in the "utility maximizing" sense) to model human decision making in a choice settings in which one person's welfare (utility) is essentially independent of the decisions of other persons. We used mathematical models to analyze consumer behavior, the theory of the firm (in competitive and monopolistic markets), decision making under uncertainty, and intertemporal decision making.

B. About two ago, we shifted to "small" number choice setting in which one person's utility (or net benefits or profit) depends in part on the decision depends in part on the decisions of other players. Most of that material focused on two person games, but we also showed that (at least in the case of lotteries) the same approach can be used to characterize the Nash equilibria of games with a large but finiete number of players.

i. We began the material on game theory with the simplest possible game settings, two person and two strategy games, and explored how different payoff structures affect the equilibrium outcomes of a variety of such games.

- For example, we demonstrated that the PD-type of payoff structure implies the existence of pure dominant strategies, and that the typical Nash equilibrium is not Pareto optimal.
- We also noted that suboptimal equilibria may also arise in assurance games.

ii. In general, whether the Nash equilibrium outcome of a particular game is Pareto Optimal or not depends on the relative sizes of the payoffs associated with each players strategies.

- These are partly determined by nature of the game (which is characterized by strategy sets and payoff functions) and partly by the strategy choices of the other players.

C. We then showed how the 2x2 game matrix can be generalized to analyze settings in which there are a more than two strategies, and more than two players.

i. For example, we extended the 2x2 game to three strategies, and then to a continuum of strategies (in games with a mixed strategy equilibrium, externality games, and lottery games). The logic of which can be applied to a wide variety of small-number settings.

ii. For example, we used the lottery game to illustrate how decisions can be made in a setting in which there are an infinite (continuum) of strategies.

- To find the equilibrium in such games, we first found the **best reply functions** of the individual players.
- Then we found the combination of strategies that occurred if all players were simultaneously on their best reply functions.

- This combination of strategies was a Nash equilibrium of the game, because the fact that every player was simultaneously on their best reply function implied that each had maximized his utility (payoff)--given the choices of the other players.
- And, we were able to character such equilibria for N-player versions of any contests that can be thought of as "lotteries."

D. The last chapter of the three game theory chapters uses game theoretic models to analyze markets with small numbers of firms.

i. In the choice settings of interest to students of industrial organization, "continuous strategy" sets exist. Firms can vary output levels or prices and are assumed to be able to select any price or quantity of interest. However, the models generally assume that firms can control only one of these variables and that market forces determine the others.

II. An Early Economic Application of Game Theory: Cournot Duopoly

A. There are three widely used models of duopoly: (1) *Cournot* (based on symmetric quantity competition), (2) *Bertrand* (based on symmetric price competition), and (3) *Stackelberg* (based on asymmetric quantity competition with a first and second mover).

i. The Cournot duopoly model is the most widely used in economics textbooks and is where we'll begin.

ii. The Stackelberg model is less widely used, but the Stackelberg modeling structure is often used to model games in which players make decisions in a sequence of some kind, and chose their strategies in part because of anticipated reactions on the part of the other players.

iii. The Bertrand model is less widely used, but should not be totally neglected, because it yields simple and direct predictions about pricing and output.

B. In the **Cournot** model (sometimes called Cournot/Nash duopoly), two firms produce identical goods and make their *output* decisions independently of one another.

i. Each firm takes the other's output given, and selects its own best output given that assumption given the *downward sloping market demand curve* for the product in question. Total output and market price are represented as equilibria to the "noncooperative" production game between the two firms.

ii. Best reply functions are used to characterize each firm's ideal output levels for given outputs of their rival. The equilibrium occurs where the best-reply functions cross. (Thus, Cournot actually invented this concept of Nash equilibrium well before Nash. In 1838.)

C. As an illustrating example of a Cournot duopoly setting, suppose that the market in question has a demand curve: $Q = 1000 - 10P$.

i. Assume also that firms are profit maximizers, sell identical products, and that profit is simply revenues from sales less the cost of producing the goods sold;

- Thus, firm "A's" profit can be written as $\Pi^A = PQ^A - C^A$.
- In order to know (or estimate) their profits they will have to know the market price. The market price depends on the total amount brought to the market by both firms, not simply that brought by firm "A."
- Demand curves "slope downward," which means that the more output is "brought to the market" the lower prices tend to be.

ii. Given our assumed demand curve, $Q = 1000 - 10P$, and the affect of total market output on market price can be written as: $P = 100 - 0.10*Q$.

- (This way of writing a demand curve is often called an inverse demand curve, because it goes from quantities into prices, rather than from prices into quantites.)

iii. If there are just two firms, A and B, then $Q = Q^A + Q^B$ and firm A profits can be written as:

- $\Pi^A = [\ 100 - 0.10*(Q^A + Q^B)\ ]\ Q^A - C^A$

iv. To simplify a bit more, let us also assume that the cost function is the same for each firm exhibits constant returns to scale and can be written as:

- $C = 5Q$
- so the profit of firm A is simply $\Pi^A = [\ 100 - 0.10*(Q^A + Q^B)\ ]\ Q^A - 5Q^A$
- or (multiplying), $\Pi^A = 100 - 0.10(Q^A)^2 - 0.10Q^AQ^B - 5Q^A$
- Notice that firm A's profit (his payoff in this game) is affected by the other player's output decision, as is typical in a game setting.

v. To find firm A's profit maximizing output, we need to find the "top" of the profit function, which can be found where the slope of the profit function is zero.

- Differentiating with respect to Q^a and setting the result equal to zero yields:

$$100 - 0.20\,Q^A - 0.10Q^B - 5 = 0$$

- Solving this "first order condition" for Q^A allows us to characterize the output level that maximizes firm A's profit for each output level that B might choose:
- $100 - 10Q^B - 5 = 0.20\,Q^A$
- $95 - 0.10Q^B = 0.20\,Q^A$
- $475 - 0.50Q^B = Q^A$
- or $Q^A = 475 - 0.50Q^B$

vi. (This equation is **the best reply function** of firm A. It tells firm A how much to produce to maximize firm A’s profits for every possible output level of firm B. There is no dominant pure strategy. The optimal response varies with the particular among brought to market by B.)

vii. A similar best reply function can be found for firm B (Do this as an exercise!)

- $Q^B = 475 - 0.50Q^A$

viii. At the Nash equilibrium, both firms will be on their best reply function--that is to say both firms will be maximizing their profits given the other firm's output.

- This implies that
- $Q^B = 475 - 0.50Q^A$ and
- $Q^A = 475 - 0.50Q^B$
- will simultaneously satisfied.

ix. Substituting for Q^B into Q^A allows us to find the Nash equilibrium levels of output for firm A. allows us

- $Q^{A*} = 475 - 0.50Q^{B*}$ which implies that $Q^A = 475 - 0.50\,(475 - 0.50Q^A)$
- multiplying through yields $Q^A = 475 - 475/2 + 0.25Q^A$
- gathering the Q^A terms and a bit of arithmetic yields $0.75\,Q^A = 475/2$
- Thus $Q^{A}** = (4/3)(475/2) = (950/3) = 316.6$
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- By symmetry (show this as an exercise) we also know that $Q^B* = 316.6$,
- which implies that total output, $Q* = 633.3$, namely, the sum of the outputs of the two firms.
- (Symmetry could also have been used to simplify the math a bit, by using the symmetry "trick" on either firm's best reply function and solving for Q**. The longer derivation is done above as an illustration of how to do so if the firms are not effectively identical.)

x. Substituting back into the profit function allows us now to determine each firm's payoff:

- The profit of firm A is simply $\Pi^A = [\,100 - 0.10*(Q^{A*} + Q^{B*})\,]\,Q^{A*} - 5Q^{A*}$
- or, substituting for the "Qs" we have $\Pi^A = [\,100 - 0.10*(633.3)\,]\,316.6 - 5(316.6)$
- A bit of algebraic arithmetic yields:
- $\Pi^A = (95)(316.6) - 63.3\,(316.6)$ or $(31.66)\,316.6 = 10{,}023.556$

D. In equilibrium both firms are simultaneously maximizing profits and, because of the symmetry assumed, both will earn the same profits from their sales of identical quantities.

- Geometrically the equilibrium occurs at the intersection of the two firm's best reply curves.

E. As a practice exercise, show that total output increases relative to the monopoly market and prices fall.

i. Recall that a monopolist will simply maximize profit, which in this case is:

- $\Pi = [\ 100 - 0.10*(Q)\ ]\ Q - 5Q$
- Find the profit maximizing output for the monopolist by differentiating its profit function with respect to Q, setting the result equal to zero, and solving for Q.
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ii. Next find the profit levels of the two firm Cournot Duopoly).

- Which industry earns higher profits?
- (Hint: the optimal output for the monopolist is Q*=475 which, of course is less than 633.3.)
- So, in which type of market are consumers better off?

F. **The Cournot model can be generalized in a number of ways**. For example, the inverse demand curve and cost function can be made more abstract and/or general. One can also extend the Cournot approach to characterize markets in which more than two firms interact.

i. This extension shows the continuum between monopoly and competitive markets in a rather neat way. It assumes quantity competition by price takers, as normally assumed in competitive markets.

ii. Consider the case in which the inverse demand curve is linear: $P = a - bQ$, $C_i = cQ_i$, and there are *n* identical firms participating in the market. Each firm makes its decision independently of one another. (That is to say, there are no cartel-like meetings or coordination of output strategies).

iii. Firm 1's profit in this case is: $\Pi_1 = a - b(\sum Qi)]\ Q_1 - c\ Q_1 = a - bQ_1^2 - b \sum_{n-1} Qi)]Q_1 - c\ Q_1$

iv. Differentiating with respect Q_1 yields:

- $a - 2bQ_1 - b\sum_{n-1} Qi - c = 0$

v. This can be solved for Q_1 as a function of parameters of the demand function, the outputs of the other firms and firm 1's cost function. Shifting the Q_1 term to the right and a little algebraic arithmetic yields:

- $Q_1^* = (a-c)/2b - \sum_{n-1} Q_i / 2$

vi. This is firm 1's best reply function and similar functions can be derived for all the other firms as well.

vii. The easiest way to find the Nash equilibrium is to assume that there is a symmetric equilibrium in which case all the firm outputs are the same.

viii. Simplifying $Q_1* = Q_2 *= \ldots Q_N *$ into the above yields:

- $Q_1* = (a-c)/2b - (n-1)Q_1*/2$
- Multiplying by 2 and gathering terms yields:
- $2Q_1* + (n-1)Q_1*= (a-c)/b$

ix. which implies that at the **symmetric Nash Equilibrium**:

- $Q_1** = (a-c)/b(n+1) = [(a-c)/b]\ [1/(n+1)]$
- **Total market output is n $Q_1** = [(a-c)/b]\ [n/(n+1)]$**

x. Substituting this into the inverse demand curve, gives us the market price:

- $P* = a - bQ = a - b\{[(a-c)/b]\ [n/(n+1)]\} = a - [(a - c)][n/(n+1)]$
- (Note that this formula works for the 2 firm case worked out above.)

xi. **Note also that as N approaches infinity n/(n+1) approaches 1 and P* approaches c. That is to say, marginal cost pricing emerges as the number of firms grows large**.

- [Do the algebra for this short proof as an exercise.]
- It can also be shown that firm profits converge to zero as N gets large. Show this as an exercise.]

G. This n-firm model of market competition is sometimes referred to as Imperfect Competition. (It is best to think of monopolistic competition as occurring among firms that produce somewhat different products, which requires a bit different modelling structure, although a similar one.)

i. There are many ways that the Cournot and imperfect competition models can be generalized.

- For example, the demand function may be generalized a bit to take account of consumer income. Suppose each firm has cost function $C = cQ$ and let the inverse market demand be:
- $P = XY - eQ$ where market production is $Q = [Q^a + (N-1)Q^o]$

ii. The first order condition that characterizes maximal profits for firm "a" is now

- $(XY - e(N-1)Q^o - 2eQ^a) = c$

iii. Firm a's reaction function is thus: $Q^a = [XY - e(N-1)Q^0 - c\,] / 2e$

iv. and the Cournot Nash equilibrium output for a typical firm at the symmetric equilibrium is : $Q^* = [XY - c] / (N+1)e$ and total output is N times as large:

- $Q^* = \{[XY - c] / e\} \{N/(N+1)\}$

v. As N approaches infinity, the total output again approaches the competitive equilibrium.

H. **Perfect Competition**, thus, is the limiting case that can be generated by entry in a Cournot-Nash type model of markets.

- In general, there are a wide variety of models of imperfect competition.
- These vary mainly with respect to the manner in which players anticipate or fail to anticipate reactions of other players in the game and also the extent to which products are similar rather than identical.
- [The expectational differences among the models developed are often referred to as "the" **conjectural variation**.]

III. The Bertrand duopoly model / Price Competition by Duopolists

A. The Bertrand duopoly model assumes that each firm **competes on price** rather than output. In effect, each firm bids for the business of consumers in a market with homogeneous products by choosing a price for its products.

- If consumers have complete information, each firm can secure essentially the entire market by charging a price just below that of its competitor.
- A sequence of profit maximizing prices offers, thus, declines, until at the Nash equilibrium, neither firm earns a positive economic profit.
- In the case where each produces via constant returns to scale, this implies that the prevailing market *price* **under this form of duopoly** *is exactly the same as that of a perfectly competitive* market.

IV. The Stackelberg Duopoly Model / An Asymmetric Form of Duopoly

A. In the **Stackelberg** model, a market is shared by two firms. However, in contrast to the Cournot model, there is a "natural" order to the game.

i. You can think of the Cournot example as representing the decisions of two farmers who independently bringing baskets of potatoes to a local market, without knowing exactly what the other does.

- At the Cournot-Nash equilibrium, each is satisfied with the quantity of potatoes that they brought to market given what the other farmer has done.

ii. In the Stackelberg case you can imagine farmer B bringing his potatoes to market after observing the quantity brought by farmer A.

- The Stackelberg case demonstrates that the order of decision making can have important affects on strategy choices and game equilibria.
- This simple two-step model, thus, represents a significant generalization of the usual simultaneous "one shot" games.

iii. In the usual Stackelberg model, one firm (game player) announces its output first (the leader) and the other announces his second (the follower).

- The first mover will try to anticipate how his or her decision will affect the decision of the second mover.
- This requires the first firm to estimate the best reply function of the second firm, which can either be substituted into the demand function or used as a constraint.

iv. The Stackelberg two-step game can be used to represent a wide variety of choice settings in which there is a "natural" first and second mover, as we will see later in the course.

- The sequential equilibrium of Stackelberg games is an early example of a **sub-game perfect equilibrium**.
- It is also an early application of the "backward induction" analytical technique.

B. A typical Stackelberg model. Suppose that there are two firms but firm a's decision will be known by firm b when it makes its output decision.

i. Firm a, knows that firm b will choose its output to maximize its profits given what firm a has done. So, it chooses its profit maximizing output with the "profit maximizing output schedule" of its competitor in mind.

ii. Firm a is said to be the "leader," although this vocabulary is used less today than it was two decades ago.

iii. The definition of profit has not changed, so: $\Pi^A = PQ^A - C^A$

- If we use the same specific functional forms that we used in the Cournot Duopoly problem we will have:
- $\Pi^A = [\ 100 - 0.10*(Q^A + Q^B)\ ]\ Q^A - C^A$
- which is exactly the form that we used above in the Cournot problem.

iv. However, in a Stackelberg model case, A does not take Q^B as given, but knows that B's behavior will be affected by his own decision about output.

- In fact, A knows that B will choose the quantity that maximizes his own profits given the quantity brought to market by A.

v. From our previous effort on the Cournot duopoly model, we know (and so does A) that B will follow the rule: $Q^B = 475 - 0.50Q^A$, as per B's best reply function.

vi. Thus, A anticipated B's response when bringing his own goods to market, and maximizes:

- $\Pi^A = [\ 100 - 0.10*(Q^A + Q^B)\ ]\ Q^A - C^A$ where $Q^B = 475 - 0.50Q^A$ and $C^A = 5Q^A$, thus,
- $\Pi^A = [\ 100 - 0.10*(Q^A + (\ 475 - 0.50Q^A\ ))\ ]\ Q^A - 5Q^A$
- $= 100Q^A - 0.10(Q^A)^2 - 47.5Q^A + 0.05(Q^A)^2 - 5\ Q^A$

vii. In order to find the profit maximizing output, we again differentiate Π^A with respect to Q^A and set the result equal to zero:

- $52.5 - 0.10\ Q^A - 5 = 0$ (in economic terms this is MR - MC = 0)
- or $Q^A = 475$
- Note that the marginal revenue function now includes the effect of the leader's anticipated effect on firm b's change in output and on the prevailing price, as well as the leader's own effect on price.

viii. To find out total market supply we now go back to firm B's best reply function and substitute for Q^A:

- $Q^B = 475 - 0.50Q^A = 475 - 0.50\ (475) = 237.5$
- So, the total output is now 237.5 + 475 = 612.5 which is between the monopoly output and the Cournot output.
- So profits will be higher under Stackelberg than under Cournot, and they will be higher for the first mover than for the second mover.
- (Calculate the profit levels for A and B as an exercise.)

C. The **Stackelberg type of model can be used to model a wide range of settings in which there is a "natural order" to the decisions** of interest or in the information sets of the players of interest.

i. For example, in crime control, normally the legal system "moves first" by imposing a schedule of fines and allocating resources to the police and courts. The criminal moves second by choosing his crime rate.

ii. Clearly the best strategy for the "legal system" (as desired by the median voter) is to take account of how the typical criminal will respond to its decision regarding penalties and policing effort.