I. Optimization, Comparative Statics, and the Implicit Function Theorems

Methodological individualism maintains that all "social" phenomena can be explained in terms of the behavior of individuals. This approach to social science is one reason why economists and other social scientists use models of individual decision making as the basis of their theories of social outcomes. For example, market demand curves are derived from individual demand curves which are themselves derived from choices that can be characterized as solutions to constrained utility maximization problems faced by individuals. Supply curves are based on decisions reached by self-interested firm owners and input owners. Together supply and demand determine market price and output. Market prices and outputs change if the average individual consumer's or firm's circumstances or expectations change, because such changes lead to different decisions by consumers and/or firms. Our models of consumer choice and firm profit maximizing aims demonstrate how and why such changes in plans may occur.

However, it is not obvious at this point whether the explanations and predictions of our models are limited to circumstances in which consumer utility functions and firm production functions can be characterized with specific families of functions such as the multiplicative exponential ones that we've used in most of our models. What we have shown so far is that many of those predictions hold for a variety of specific parameters of those functional forms. Thus, similar prediction tend to hold for any utility and production functions drawn from the families of functions that we've been using in our models. That has been shown in all of our derivations that used abstract characterizations of choice setting parameters (e.g. by representing exponents, prices, costs, etc, as letters rather than specific numbers).

To claim greater generality requires more general approaches to modelling consumer and firm behavior and of the effects of market structure on market prices.

In this final chapter of the course, even more general tools for analyzing how the optimizing choices of individuals and firms are developed. It shows how similar, but more general analyses, can be undertaken using even more general families of functions. For example, most of our previous results hold for any continuous strictly concave utility, production, or payoff function. To show this requires introducing 3 new results from mathematics: the sufficiency theorem, the implicit function theorem, and the implicit function differentiation rule.

However before doing so, it is useful to review methods for thinking about comparative statics using explicit function forms. We have done a bit of this during the course, but for the most part we've been content to characterize equilibrium results of various kinds, rather than focusing on how various "shocks" can change those equilibria.

I. A Short Review of Comparative Statics

What micro economists refer to as "comparative statics" is essentially the study of how changes in parameters of individual and firm choice problems affect their choices and thereby market and other social outcomes. The mathematics that describes how changes in parameters (wealth, prices, tastes, technology) change decisions in models based on concrete functional forms can be determined by differentiating what might be called "behavioral" or "end result" equations (demand, supply, cost, best reply, or Nash equilibrium) with respect those "exogenous variables.

For the most part the course to this point has focused on "behavioral" equations that explain or predict how consumers or firms respond to price (individual demand and supply curves) and how game players respond to minor changes in the game (as with different sized prizes). Those effects on behavioral equilibria are exercises in **comparative statics**, although we did not call them that. That term generally is used for studies of factors that can change equilibrium outcomes. For example, they may explain why prices in a competitive markets change as consumer income increases, or as input prices paid by firms increase, or as the prize changes in an innovation contest.

We have already done quite a bit of that in the course, but we have not emphasized that what we were doing was comparative statics.

An Illustration: Abstract Concrete Functional Forms and Comparative Statics

A. In the first third of the course, we found that an individual with a Cobb-Douglas utility function will have a **demand function** of the form: $X_1^* = aW/P_1$ when he or she has a utility function of the form $U = X_1^a X_2^{(1-a)}$ and $W = P_1X_1 + P_2X_2$.

- We used the Lagrangian technique to derive this demand function.
- We formed a Lagrangian function: $L = X_1^a X_2^{1-a} - \lambda\,(W - P_1X_1 - P_2X_2)$

- This was differentiated with respect to the control variables X_1, X_2, and the Lagrangian multiplier λ.
- The first two first-order conditions for X_1 and X_2 were divided to eliminate the lambda terms and did a bit of algebra yields $X_2 = [(1-a)/a]\ [P_1/P_2]X_1$
- This result was substituted into the budget constraint (for X_2) (the third first order condition) and that function was solved for X_1 as a function of the other variables which gave us the demand function for X_1: $X_1^* = aW/P_1$.

i. Given that result, we can see that as consumer wealth increases, such consumers will **always** increase their demand (e.g. the demand curve shifts to the right).

- $dX_1^* / dW = a/P_1 > 0$ for all $a>0$ and $P_1>0$.
- In geometric terms, this consumer's demand curve shifts out to the right as his or her wealth increases.

ii. This is an elementary form of comparative statics on consumer demand.

- If all consumers are basically similar (or at least have Cobb-Douglas utility functions, the effect of an increase in wealth will either be the same for each consumer or qualitatively the same for each (if the exponents differ), which implies that the market demand curve for such consumers increase (shifts right) as wealth increases.
- Note that the mathematics of Cobb Douglas utility functions (and similar exponential multiplicative functions) implies that all goods are normal goods, if consumers have utility functions with positive exponents? (Explain why.)

iii. When the consumer modeled is the "typical" or "average" consumer, his or her demand function can be used as the foundation for a theory of market demand. The market demand function in that case is simply the average consumer's demand function multiplied by the number of consumers in the market of interest (N).

- In the above case, **the market demand function** (curve) is $NQ^* = N\ aW/P_1$ where a and W are specific values for the average consumer.

B. **A supply function (curve)** for a typical firm in the market of interest can similarly be used as the foundation for a market supply curve. A firm's supply curve can be derived from its cost function.

i. An example of such a cost function is $C = Q^c wr$ where w is the prevailing wage rate, r is the rental cost of capital or interest rate, and $c > 0$.

- Profit is $\Pi = PQ - C = PQ - Q^c wr$
- Differentiating with respect to Q and setting the result equal to zero provides the first order condition for maximizing profit (assuming that the profit function is strictly concave, e.g. as when that function has positive first derivatives and negative second derivatives).
- $P - cQ^{c-1}wr = 0$

- Solving for Q* yields this firm's supply curve
- $Q^* = [P/cwr]^{(1/c-1)}$

ii. This firm-level supply function, in turn, can be used as the basis for a **market supply curve** if the firm modeled is "typical" or "average" and we know the number of firms in the industry (M). In that case, **the market supply function is simply** $MQ^* = M[P/cwr]^{(1/c-1)}$.

C. Given market demand and supply functions, we solve for the equilibrium price and quantity.

i. Given our results from the above, this requires **finding the price where the quantity demanded equals the quantity supplied**, which requires: $N\ aW/P_1 = M[P/cwr]^{(1/c-1)}$

ii. As before, this is a matter of algebra at this point. We want to solve for P (which in this case is the price that we refer to as P*--explain why).

- Multiply both sides by P and then factor the righthand side to create:
- $NaW = MP[P/cwr]^{(1/c-1)} = M\ [P]^{(1+1/c-1)}\ [1/cwr]^{(1/c-1)} = M\ [P]^{(c/c-1)}\ [1/cwr]^{(1/c-1)}$
- Isolate the P term by dividing both sides by M and multiplying both sides by $[cwr]^{(1/c-1)}$
- $[P]^{(c/c-1)} = NaW[cwr]^{(1/c-1)}\ /\ M$
- Taking the $(c/c\text{-}1)^{th}$ root of both sides yields an expression for the **market clearing price**.

iii. $P^* = \{NaW[cwr]^{(1/c-1)}\ /\ M\}^{((c-1)/c)}$

- Assuming the c>1, this implies that all the terms in the numerator unambiguously increase the market clearing price and the term in the denominator (M) unambiguously reduces the equilibrium price in this market.
- Any increase in consumer wealth (W) or in preferences for good 1 (a) would increase market prices as would an increase in the marginal cost of production (c, is positively related to marginal cost).
- An increase in the number of consumers also increases market prices, whereas an increase in the number of firms decreases equilibrium prices in this market.

iv. That is what model-based comparative statics entails. The results apply for any function that belongs to the family of utility functions and family of cost functions used.

- **To the extent that those functional families are "realistic" the results will apply to a wide variety of markets in which consumers and firms are price takers.**
- Notice that most of these effects can be traced back to shifts in either the individual demand function (and its associated curve) or shifts in the individual firm supply function (and its associated curve).

D. Very similar results can often be obtained from more general models and mathematical tools for analyzing the effects rational decision making on consumer and firm choices. Changes in the same variables will often have similar effects on equilibrium outcomes, as we will see in the next sections of this chapter.

II. More General Models of the Behavior of Optimizing Individuals and Organizations

A. More general models of the choices of firms, consumers, and others require a few additional results from mathematics.

B. The three most useful are **the sufficiency theorem**, the **implicit function theorem**, and the **implicit function differentiation rule**.

C. The **sufficiency theorem** demonstrates that:

v. **If an objective function is concave and the associated feasible set is convex, the first order conditions of an optimization problem completely characterize the optimum values of all control variables.**

- It bears noting that is true whether we can solve for "nice" closed form solutions for the first order conditions or for the market phenomena of interest.
- Moreover, this theorem is also true if we do not know the exact functional form of the relationships of interest, or do not want to make specific assumptions about functional forms.

vi. In many cases, it is EASIER to develop models with abstract functional representations of the relationships of interest than with concrete functional forms.

- Comparative statics can also, perhaps surprisingly, be undertaken using very abstract functional forms.

D. In order to show how to undertake this more abstract modeling, we require two additional tools: namely the implicit function theorem and the implicit function differentiation rule.

E. The **implicit function theorem** allows a variety of relationships to be deduced from the first order conditions, including ones that are useful for comparative statics.

i. The **implicit function theorem** implies that the first order conditions to be used: to characterize the solution (optimal value of the control variable(s)) as a function of the parameters of the optimization problem.

ii. **The Implicit Function Theorem** (see Chiang 205 - 206, La Fuente 5.2):

- Given a function such that: $F(Y, X_1, X_2, , , X_m) = 0$
- Where the function F has continuous partial derivatives $F_Y, F_{X1}, F_{X2}, \ , \ , F_{Xm}$,

- and at point (Y^o, X^o_1, X^o_2, X^o_3, , , X^o_m) satisfying condition i, F_Y is nonzero,
- *In that case,* there exists an m-dimensional neighborhood, N, of the point (Y^o, X^o_1, X^o_2, X^o_3, , , X^o_m),

  in which an implicit function exists that characterizes each of the "ideal" values of the control variables of F as a function of non-control variables (the "parameters" of the choice problem): as, for example, $Y = f(X_1, X_2, X_3, , , X_m)$

iii. This function satisfies $Y^o = f(X^o_1, X^o_2, X^o_3, , , X^o_m)$ in particular, and more generally, $Y = f(X_1, X_2, X_3, , , X_m)$ for all points within the neighborhood.

- This gives the function the status of an identity within neighborhood N.
- Moreover, the implicit function, f, is continuous and has continuous partial derivatives with respect to X_1, X_2, , , X_m.

iv. **In microeconomic applications**, the "zero function," $F(Y, X_1, X_2, , , X_m) = 0$, is usually the first order condition of some optimization problem, and the implicit function is the individual's demand function, a firm's supply function, or a game player's best reply function.

- However, the theorem applies to ANY function that equals zero.
- This includes cases in which one has used concrete functional forms, but no closed for solution for Y as a function of all the Xs can be worked out.
- The N-equation version of the implicit function theorem is broader in scope but essentially similar. (See Chiang 210 - 211, la Fuente 5.2.)

v. **In cases where there is a single first order condition, there is a fairly straightforward method by which partial derivative of the "implicit function" can be computed.**

vi. **The implicit function differentiation rule is used to characterize the partial derivatives of the implicit function "generated" as above.**

- The rule is surprisingly simple (la Fuente, theorem 2.1).

The partial derivative of implicit function $Y = f(X_1, X_2, \ldots F_N)$ with respect to X_i is simply:

$\mathbf{Y_{Xi} = F_{Xi} / - F_Y}$ (where subscripts denote partial derivatives with respect to the variable subscripted).

F. The partial derivatives of such an implicit function are obtained using the implicit function differentiation rule. That rule can be **derived from** the total derivatives of F.

i. Recall that $F (Y^o, X^o_1, X^o_2, X^o_3, , , X^o_m) = 0$

- Thus, the total derivative of F has to add up to zero.

$$dY\ F_Y + dX1\ F_{X1} + \ldots..\ dXm\ F_{Xm} = 0$$

ii. Consequently, if we allow only Xi and Y to vary,

$$dYF_Y + dXi\ F_{Xi} = 0$$

- Solving this expression for dY/dXi yields:

$$dY/dXi = F_{Xi} / - F_Y$$

iii. The implicit function differentiation rule allows one to characterize how the solution to an optimization problem varies as parameters of the problem vary.

G. Illustrating Example (1): Properties of an individual firm's "abstract" supply function

i. Suppose that "Acme" has a cost function C = c(Q, w, r, t), where w is the average wage rate, r the average cost of capital and t is technology. Suppose that C is twice differentiable with positive first derivatives for Q, w, and r, but a negative first derivative for t. Suppose also that Acme sells its product in a competitive market where each unit of its output can be sold at price P.

ii. In this case, Acme's profit function is: $\Pi = PQ - C$

- Note that if C's second derivatives are positive, the profit function will be strictly concave.
- This allows the sufficiency theorem to be employed.

iii. Differentiating Acme's profit function with respect to Q and setting the result equal to zero yields: $P - C_Q = 0$

- Note that this satisfies the requirements of the implicit function theorem.
- Thus, Acme's supply, Q*, can be written as Q* = s(P, w, r, t)
- Note also that P, w, r, and t are parameters of Acme's choice problem. They are exogenous variables as far as Acme is concerned.

iv. We can use the implicit function differentiation rule to determine how changes in Acme's choice setting affect is choice of output to produce and sell. We can, for example, determine the slope of Acme's supply function.

- Let $H \equiv P - C_Q = 0$ at Q*
- Then $dQ^*/dP = H_P/-H_Q$
- H_P is the derivative of $P - C_Q$ with respect to P which is just 1, because P appears in only one place. It is not part of the cost function.
- H_Q = is the derivative of $P - C_Q$ with respect to Q which is just $-C_{QQ}$, the negative of the second derivative of the cost function.

- Thus $H_P/-H_Q = 1/-(-C_{QQ}) > 0$
- Recall that $C_{QQ}>0$ in order to assure the concavity of the profit function. The double negative implies that the "sign" of $H_P/-H_Q$ is greater than zero.
- Thus, Acme's supply curve slopes upward.

v. The effects of a change in wage rate on supply can be determined in a similar way.

- $dQ^*/dw = H_w/-H_Q$
- H_Q is the same as before $= -C_{QQ}$
- H_w is the derivative of $P - C_Q$ with respect to w. Recall that the marginal cost function (C_Q) includes all the variables as the cost function.
- The notation for the derivative of marginal cost with respect to wages is C_{Qw}
- So, $dQ^*/dw = H_w/-H_Q = -C_{Qw} / -(-C_{QQ})$
- We have not made assumptions about the "cross partials" of the cost function, but logically those with respect to w and r are both greater than zero (marginal costs increase as wage rates or the cost of capital increases) while the cross partial with respect to technology is most likely less than zero.
- Given this, $dQ^*/dw = H_w/-H_Q = -C_{Qw} / -(-C_{QQ}) < 0$
- The double negatives of the denominator assure that it is positive. The negative of the cross partial of production costs with respect to wages ($C_{Qw}>0$) is negative.
- So, the overall term is negative. Acme's supply curve shifts to the left when wages increase.

H. **Illustrating Example (2): Properties of an individual's "abstract" demand function**

i. Suppose that "Al" has a utility function, $U = u(X_1, X_2)$ which is monotone increasing in X_1 and X_2, twice differentiable and strictly concave. The latter may be assured by assuming that: $U_{X1\,X2} \geq 0$, $U_{X1\,X1} < 0$, and $U_{X2\,X2} < 0$.

ii. Al wants to find the utility maximizing combination of X_1 and X_2 given the budget constraint that he faces, $W = P_1X_1 + P_2X_2$.

- Note that the budget constraint implies that $X_2 = (W-P_1X_1)/P_2$
- Substitute this into the utility function for X_2: $U = u(X_1, (W - P_1X_1)/P_2)$
- Differentiate with respect to X_1 which yields: $U_{X1} + U_{X2} (- P_1/P_2) = 0$

iii. The value of X_1 that satisfies this first order condition will maximize utility.

- Denote such that value of X_1 as X_1^*

iv. Note that at X_1^*, the first order condition is a function like F in the definition of the implicit function theorem; that is to say, the "foc" always equals zero at X_1^*.

- Since the first order condition is differentiable (remember that we assumed that U was twice differentiable), an implicit function exists that characterizes X_1^* as a function of the other parameters of the choice problem.

$$X_1^* = x(W, P_1, P_2)$$

- Economists refer to this function as Al's **demand function** for X_1.

v. The effect of a change in the price of good 1, P_1, on Al's demand for good 1 can be characterized using the implicit function differentiation rule:

$$X_1^*{}_{P1} = F_{P1} / - F_{X1}$$

vi. Given our first order condition in equation iv above, $F_{P1} / - F_{X1}$ can be written as:

$$F_{P1} / - F_{X1} = \frac{[\, U_{X1X2}(-X_1/P_2) + U_{X2}(-1/P2) - U_{X2X2}(P_1/P_2)(-X_1/P_2)\,]}{-[U_{X1X1} + 2\,U_{X1X2}(-P1/P2) + U_{X2X2}(-P_1/P_2)^2\,]}$$

This expression is determined by carefully calculating the derivatives of the first order condition, $U_{X1} + U_{X2}(-P_1/P_2)$, with respect to P_1 and X_1.

vii. Here is briefly how this expression was found. (**Read though this carefully, it is important**!)

- First recall that the first order condition was: $U_{X1} + U_{X2}(-P_1/P_2) = 0 \equiv F$
- The implicit function theorem implies that the solution (here for X_1^*) can be written as an implicit function of the other variables in the first order conditions:
  $X_1^* = f(W, P_1, P_2)$
- Notice that the "W" is not obviously in the first order condition; you have to remember that each partial derivative is a function that includes all the variables of the original objective function (the parent function). So, for example, U_{X1} is a function that includes X_1 as its first argument and $(W - P_1X_1)/P_2$ as its second argument.
- Thus, you could write $U_{X1} = f'(X_1, (W - P_1X_1)/P_2)$, although we don't usually bother to do so. This is why "W" is included as a variable in the implicit function.
- This is also true of function U_{X2}, which could be written as $U_{X2} = g'(X_1, (W - P_1X_1)/P_2)$, although we don't usually bother to do so.
- These terms and their derivatives are nonetheless important.

- They play an important role in finding the various derivatives of function F, which are really just derivatives of the first order condition (foc).
- The steps to find F_{P1} (the numerator) are the following.
  - The first term of the foc. includes the price of good 1 only in its second argument so there will be just one term in its derivative with respect to the price of good 1.
  - The derivative of U_{X1} with respect to its second argument (X_2) is written as U_{X1X2}
- In this case, we are using the composite function differentiation rule: the derivative of Y = h(z(x) is (dh/dz)(dz/dx) –the derivative of the "outside function" times the derivative of the inside function. In the case we are interested in this is **$(U_{X1X2})(-X_1/P_2)$** which is again the derivative of the outside function times the derivative of the inside function. This is the first term in the numerator of the complex expression for $F_{P1}/ - F_{X1}$.
- The derivative of the second term in the f.o.c. is a bit more complicated in that we have to use both the product rule and the composite function differentiation rules. The product rule says that if Y = g(x)h(x), then Y_x = (dg/dx) h(x) + g(x)(dh/dx). Recall that P_1 appears in the second argument of U_{X2} for the same reason that it was in the second argument of U_{X1}.
  - The derivative of U_{X2} with respect to X_2 is written as U_{X2X2} so the composite differentiation rule yields U_{X2X2} $(-X_1/P_2)$ which is then multiplied by the other part of the multiplicative term we are differentiating to get: **U_{X2X2} $(-X_1/P_2)$ $(- P_1/P_2)$**. This is the last term in the numerator of the quite complex expression for $F_{P1}/ - F_{X1}$.
  - There is another term from the product rule. It is the derivative of the second term in the product times the first term in the product, here: **U_{X2} $(-1/P_2)$** . Note that this is the middle term in the numerator of the complex expression for $F_{P1}/ - F_{X1}$ above.
- Note that H_{X1} (the denominator in the implicit function differentiation rule ($X_1{}^*{}_{P1}$ = $H_{P1}/-H_{X1}$) is just the **second derivative of the original objective function** and will have a value that is less than zero at X_1* if U is strictly concave as assumed in order to use the sufficiency theorem.
- The implicit function differentiation rule often looks simple initially, but deriving the exact functional form of the derivatives of implicit functions often require many steps, as in the demand curve derivation here.
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viii. The "sign" of the derivative of our implicit function with respect to selling price tells us whether Al's demand curve slopes downward or not.

- The "sign" is jointly determined by all the partial derivatives in the expression above.
- Most of these have already been characterized by our assumptions about Al's utility function (strictly concave) and his or her budget constraint (linear).

- From the original characterization of U we know that all of the first partial derivatives are positive
- We also know that all of the second derivatives are negative (This implies that both X1 and X2 are goods that exhibit diminishing marginal utility.).
- Along with these assumptions, positive cross partials are sufficient (but not necessary) condition for the utility function to be strictly concave. (In such cases, an increase in good 2 increases the marginal utility of good 1, and vice versa).

- **Note that together these characteristics of U imply that Al's demand curve is downward sloping**, $X1^*_{P1} < 0$.

ix. Note that the denominator of the implicit function differentiation rule (here, $-F_{X1}$) is the **second derivative** of the original objective function, and will be negative if the objective function is strictly concave.

- Thus, the qualitative effect (the sign) of a change in price on Al's optimal purchase of X_1 is determined by the numerator—although the absolute effect requires knowing values for both the numerator and denominator.
- Note that the assumptions about the utility function, positive first derivatives, negative second derivatives and positive cross partials assure that $F_{X1} < 0$, which implies that $-F_{X1}$ is a positive number.
- (As an exercise, compute the derivative of the demand function with respect to Al's wealth and see whether "positive cross partials" also rule out inferior goods.)
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I. Illustrating example (3): An Abstract Contest between 2 players

i. Suppose that there is a contest for prize R and that the probability of winning the prize rises with one's efforts or investments in the contest (E_1) and falls with the efforts of the others (E_2). In this case $P = p(E_1, E_2)$ with a positive first derivative for the first term and a negative one for the second. Suppose that the cost of one's own effort increases with one's effort and opportunity cost wage rate, $C = c(E_1, w)$.

- The expected reward of competing in this contest is thus: $R^E = PR\text{-}C$

ii. Player 1 (Al's) best reply function can be characterized by differentiating R^E with respect to E_1 and setting the result equal to zero.

- $P_{E1}\, R - C_{E1} = 0 \equiv H$

iii. The implicit function theorem implies that Al's best replay function can be characterized as:

- $E_1^* = e(E_2, w)$

- Note that E_2 is a term in the marginal probability of winning function (P_{E1}). Al's opportunity cost wage is from his/her marginal cost function (C_{E1}). Remember that both these functions include all the variables that were in their original "parent" functions.

iv. The slope of Al's best reply function is $dE_1{}^*/dE_2 = H_{E2}/-H_{E1}$

- In this case $H = P_{E1} R - C_{E1}$
- The numerator of $H_{E2}/-H_{E1}$ includes just one term: $P_{E1E2}R$ because E_2 only appears in the probability function and only in one place in that function.
- The denominator has two terms $-(P_{E1E1} R - C_{E1E1})$
- E_1 appears in both the probability function and cost function, but only in one place in each of these functions.
- So, $H_{E2}/-H_{E1} = P_{E1E2}R /-(P_{E1E1} R - C_{E1E1})$
- To determine whether the slope rises with the opponent's effort (Mr. or Ms. 2) or falls with that effort requires thinking about these derivatives.
  - The denominator is the second derivative of the expected prize function which will be negative if that function is strictly concave. Note that this will be true if $P_{E1E1} < 0$ and $C_{E1E1} > 0$, both of which are consistent with economic intuitions about diminishing marginal returns. In that case the terms inside the parentheses are negative and the denominator is positive.
  - The cross partial in the numerator (P_{E1E2}) is likely to be negative—at least this is its most intuitively plausible sign. As player 2's effort increases, the marginal effort an increase in effort by player 1 tends to decrease. (Obviously, it would be easiest for player 1, if the other player invested no effort in the contest. The more effort player 2 invests, the less likely it is that player 1 will win.)
- With these other assumptions (some of which simply help assure that the sufficiency theorem can be applied) $H_{E2}/-H_{E1} = P_{E1E2}R /-(P_{E1E1} R - C_{E1E1}) < 0$
  - The numerator is negative and the denominator is positive, so the over all term is negative.

v. At the equilibrium, both players will be simultaneously on their best reply functions.

- Thus an equilibrium can be said to occur whenever:
- $E_1{}^{**} = e(E_2{}^{**}, w)$ and
- $E_2{}^{**} = e(E_1{}^{**}, w)$

J. The implicit function theorem and differentiation rule have a wide range of applications in economic models and in game theory.

- A super majority of theory papers for the past 30 years use these methods, although, surprisingly, explicit functional forms have been making a comeback for the past 10 years or so.

- Every model that we've developed in the class can be redone in these more abstract and general terms, and most of the results will be similar (the signs of the comparative statics will be the same) in most cases.

- Doing so would help you to internalized the logic of this more abstract and general approach to modeling individual behavior and social outcomes.

- (There are also multi-equation version of the theorem and differentiation rule, as developed in Chaing 8.5. These generalizations, however, are not very widely used and are beyond the scope of this course.)

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K. A Few Practice Problems

(1) Determine the first order conditions that characterize each of the following firm's profit maximizing level of output, Q*, in the market setting characterized. Then write solve for or characterize the firm's supply function.

i. Each firm produces its output at average cost C and sells its output at price P:

- $C = w(Q^2 + 1)$ where $P = P^o$ ("w" is the wage rate)
- $C = c(Q, w_1, w_2, r, t)$ where $P = P^o$ (Cost function c is monotone increasing and strictly *convex,* e.g. has positive second derivatives. "r" is the rental cost of capital.)
- $C = c(Q, w_1, w_2, r, t)$ and $P = p(Q,Y)$ (The cost function, c, is as before, and inverse demand function p is decreasing in Q and increasing in consumer income Y.)

ii. Use the implicit function theorems on first order conditions of part A to:

Write the ideal levels of output as a function of parameters of each firm's choice problem.

Obtain derivatives of profit maximizing output levels with respect to wage rate w_2.

What, if anything, do your answers imply about the short run effect of new labor laws that increase wage rates in the industry of interest?

Appendix I: Another Useful Theorem: Comparative Statics and the Envelop Theorem

A. Another type of comparative static result involves how changes in parameter of the choice problem (price of a substitute or input, wealth, change in technology, etc.) affects the best result (maximal utility, profit, etc.) of the decision maker.

Many such properties seem intuitively obvious and often the math is fairly easy as well.

B. **The Envelop Theorem** implies that derivatives of the "optimized objective function" are far simpler than one might expect.

- The optimized objective function (maximal utility, maximal profit, etc. functions) are simply the original objective function with the optimal values of the control values substituted in.
- Normally these optimal values are themselves functions of parameters of the choice problem, as above with our demand functions.
- Suppose that the objective function is: $O = o(X_1, X_2, T)$ and there is a constraint that states that $c(X_1, X_2, Z_1, Z_2, Z_3) = 0$.
- ["O" might be utility, X_1 and X_2 quantities of goods that the consumer can choose, and the constraint might be an abstract representation of a budget constraint, with the Z's being relevant prices and consumer income. T would be a taster parameter of some kind.]
- Suppose that you have solved the constrained optimization problem using either the substitution or Lagrangian technique and found that $X_1^* = f(Z_1, Z_2, Z_3, T)$ and $X_2^* = g(Z_1, Z_2, Z_3, T)$, after taking partial derivatives, setting them equal to zero, and using the implicit function theorem.
- The optimized objective function will be $O^* = o(X1^*, X2^*) = o(f(Z_1, Z_2, Z_3, T), g(Z_1, Z_2, Z_3, T))$
- The derivative of O* with respect to T (a parameter of the original optimization problem, such as the price of a substitute or input) will be:
- $O^*_T = O_{X1}\, X1^*_T + O_{X2} X2^*_T + O_T$

- (This derivative, O^*_{Z1}, tells you how the objective (utility, profits, etc.) is affected by changes in Z_1 given that the decisionmaker is rational and is responding optimally to changes in Z_1.)

C. This simplification is often useful mathematically and also often produces insights that are somewhat counter intuitive.

- The simplification occurs because many parts of the derivative will turn out to be terms from the first order conditions used to characterize the solutions of (or solve) the original optimization problem.

- Many of the terms in the derivatives of O* thus will be zero along the "envelop" and so disappear from the derivative (e.g. they will necessarily have a value of zero when the control variables take their optimal values").

- Normally--but not always--only the direct effects will remain. In such cases, the indirect effects--the terms including derivatives of X_1* and X_2*--turn out to equal zero or add up to zero along the "envelop," and only terms like O_T will remain.

- In such cases, $O^*_T = O_T$, which implies that the effects of other "second order" adjustments are irrelevant for (small) changes in the parameter (T) of interest.

  - See Chaing and Wainwright 15.5 for a nice illustration of the "envelop effect" for a firm's profit function.
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D. Illustration: the comparative statics for a classic consumer choice problem

i. Suppose that Al choose X_1 and X_2 to maximize $U = u(X_1, X_2)$ subject to $W = P_1X_1 + P_2X_2$.

Solve for X_2 as a function of X_1, etc. along the constraint, $X_2 = [W - P_1X_1]/P_2$

Substitute for X_2 in the utility (objective) function: $U = u(X_1, [W - P_1X_1]/P_2)$

ii. Differentiate with respect to X_1 (the only control variable left) and set the result equal to zero to form a first order condition.

- $U_{X1} + U_{X2}(-P_1/P_2) = 0$

Appeal to the implicit function theorem to characterize the demand (optimized value) for X_1.

- $X_1^* = f(P_1, P_2, W)$

- Note that the implicit function differentiation rule can be used to characterize the properties (slopes) of Al's demand function for X1.

- For example, $X_1{}^*{}_{P1} = [\ U_{X1X2}(-\ X_1/P_2) + U_{X2X2}(-P_1/P_2)\ (-\ X_1/P_2) - U_{X2}/P_2\ ]\ /\ [-z]$

- with $[z] = U_{XX11} + U_{X1X2}(-\ P_1/P_2) + U_{X2X1}(-\ P_1/P_2) + U_{X2X}(-P_1/P_2)^2$

- [z] has to be negative if there is a unique solution (e.g. if the function with the substituted terms is strictly concave) [why?] **and if the second derivatives of U are negative and the cross partial are positive, this will be the case**. [Go through [z] term by term to see that this is true.]

- [z]< 0 implies that [-z] > o, so the sign of the derivative is characterized by the numerator.

- Note that given our assumptions, every term in the numerator is less than zero, so Al's demand curve is downward sloping! that is $X_1{}^*{}_{P1} < 0$.

iii. (This is not important for the envelop function illustration, but it shows how one can use the tools of this chapter to characterize and analyze consumer demand.)

iv. **(Continuing the illustration)** Substitute X_1* into the solution above for X_2 to characterize X_2*.

- $X_2{}^* = [W-\ P_1X_1{}^*]/P_2$

Substitute X1* and X2* into either utility function above to characterize the maximum feasible utility for Al in this setting.

- $U^* = u\ (\ X_1{}^*,\ [W-\ P_1X_1{}^*]/P_2\ )$

v. The **envelop theorem can now be illustrated** by take the derivative of this function with respect to one of the parameters of the choice problem (W, P_1, or P_2).

- For example, the derivative with respect to W is:

- $U^*{}_W = U_{X1}X1^*{}_W + U_{X2}(-P_1/P_2)X_1{}^*{}_W + U_{X2}/P_2$

  - (Note that this is not a first order condition, so we do not set it equal to zero.)

- We can reorganize this a bit by factoring the $X_1{}^*{}_W$ from the first two terms:

- $U^*{}_W = X1^*{}_W\ [U_{X1} + U_{X2}(-P_1/P_2)\ ] + U_{X2}\ /P_2$

- Note that the term inside the brackets is always zero if we have maximized utility with respect to X_1 and X_2.
- [Why? Look at the derivative under "iv" above.]

- Thus, the whole first term is zero along the "envelop" so,

- $U^*_W = U_{X2} / P_2$ in this case.

vi. Thus, the increase in utility generated by an increase in wealth is equal to the marginal utility of using the additional wealth to purchase ideal amounts of either good.

- [Note that the first order condition implies that $U_{X1}/P_1 = U_{X2}/P_2$.]