I. A Few More Illustrations of Nash equilibria in 3x3 games

A. The Regulatory Dilemmas of Neighboring Governments

B. Race to the Bottom. Suppose that are two communities that are interested in regulating some activity within their own territory.

i. Suppose further that regulations in each community affect each other's prosperity, with the community with the "weakest" regulations being somewhat more prosperous than the community with the stronger community.

ii. To simplify a bit, assume that there are just three types of regulations that can be imposed: weak, medium, and strong regulations.

iii. Suppose also that the joint ideal is "medium, medium"

iv. However, the effect of local regulations (relative to that of the other community implies that each community is a bit better off weakening its regulations, given the other's regulation of the activity of interest.

v. Such games have a Nash Equilibrium in Pure strategies that is not Pareto Efficient.

vi. This "regulatory dilemma" is sometimes called the "Race to the Bottom," because each government has an incentive to under regulate the phenomena of interest (say air pollution).

The Race to the Bottom Dilemma **Community B's Environmental Regulations**			
	weak	medium	strong
A's env regs weak	A,B 6,6	A,B 8,4	A,B 9,2
medium	4,8	7,7	8,5
strong	2,9	5,8	6,6

vii. Notice also that a voluntary agreement to move to (medium, medium), as with a treaty, may not solve the dilemma because it is not a Nash equilibrium. Both players have an incentive to cheat (renege) on the agreement, because 8>7.

- It is for this reason that treaties often, although not always, have little or no effect on international air pollution [See, for example, papers by Murdoch and Sandler (1997)].
- This may be difficult to arrange in an international setting although it can be done within a federal system by higher levels of government.
- In a federal system, such problems can, however, be solved if higher levels of government punish (fine) communities that are free riding.

C. NIMBY. Now suppose that the inter-community externality in the opposite direction. That is to say, suppose that the community with the weaker regulation attracts undesirable (say, noisy, ugly, or polluting industries) into the community.

i. Assume again that there are just three levels of regulation and that the two community ideal is (medium, medium) as in the previous example.

ii. In this case, each community is just a bit better off if it has somewhat tougher regulations than its neighbor.

The Race to the Top Dilemma NIMBY Community B's environmental Regulations			
	weak	medium	strong
A's env regs	A,B	A,B	A,B
weak	6,6	4,8	2,9
medium	8,4	7,7	5,8
strong	9,2	8,5	6,6

iii. We can just slightly modify the payoffs of the above game to illustrate the new problem.

iv. This game also has a Nash Equilibrium with dominant strategies that is not Pareto Optimal.

v. This regulatory dilemma is sometimes called the "race to the top" or NIMBY (not in my backyard) problem. Each community has an incentive to tighten its regulations to prevent the annoying facility from being placed in their own community, although both would benefit if such a facility is built. At the Nash equilibrium, both communities adopt strong regulations that block the facility. However, a Pareto superior move exists at the equilibrium (7,7) is Pareto superior to (6,6).

D. **A Game with a continuum of strategies** and the possibility of **Mixed Strategy Equilibria**

i. There are many games in normal form that lack a Nash equilibrium in pure strategies.

ii. Such games fail the "double underline" test in that there is no case where the best replies of both players take them to the same cell. In other words, there are best-reply functions in pure strategies (specific choices among the strategies) that never intersect one another.

- This result is most common in games with discrete strategy sets, because "discreteness" allows one of the best reply functions to in sense "jump over" the other in a way that is less likely in games where a continuous range of possible strategies exist.

iii. The notion of a mixed strategy shifts the domain of play from a discrete set of strategies to a continuum of probabilities across the strategies.

- In the context there is always an intersection of the best-reply functions of two individuals. (There is a proof of this, which interested students can look up, but we'll not cover that in this course.)

iv. A good illustrating example is the children's game of paper, rock, scissors, where two players choose strategies simultaneously. They payoffs can be thought of as +1 if one wins, 0 if there is a tie, and -1 if one loses. With paper winning over rock, rock beating scissors, and scissors beating paper.

- The Payoff below and to the right illustrates the payoffs to such a game.
-

- (Believe it or not, there are international tournaments in paper, rock, scissors.)

The Paper, Rock, Scissors Game Bob's choice			
	Paper	Rock	Scissors
Al: Paper	A,B 0, 0	A,B 1, -1	A,B -1, 1
Al: Rock	-1, 1	0, 0	1, -1
Al: Scissors	1, -1	-1, 1	0, 0

v. Notice that when the best replies are underlined, there is no cell with two underlines. Thus, there is no Nash equilibrium in pure strategies for this game.

- In this game, the Nash equilibrium involves "mixed strategies" where each player uses each strategy with a probability of 1/3.
- Such probabilistic strategies are called "mixed strategies," because one "mixes" the available strategies rather than playing a particular response to the play of others in the game.

vi. The equilibrium is called a "**mixed strategy equilibrium**" or Nash equilibrium in mixed strategies.

- Note that there are an infinite number of possible mixed strategies in every game, but usually just one equilibrium pair of strategies from which no person can increase their expected (average) payoff given the play of the other.
- There is a mathematical proof that such an equilibrium exists for every game in normal form (e.g. matrix form). The appendix to this chapter sketches out that proof and directs you to references where that proof is worked out in somewhat more detail.
- It is sometimes said that an equilibrium in pure strategies is a special case of that proof—name one where the probability of using one of the strategies is 1 and of using the others, given the choices of other players, is 0.

II. PD-like Games with Continuous Strategy Options

A. There are many other settings in which players strategies are not discrete, but rather lie along a continuum of some sort.

- Players on a team may work more or less.
- More or less of a public good may be provided.
- One may purchase more or less lottery tickets.

B. Such games are represented mathematically by specifying a payoff (or utility) function that characterizes each player's payoffs as a function of the strategy choices of all the players in the game of interest.

C. For example, supposes there are two players that attempt to maximize a net benefit function with a Cobb-Douglas benefit function in a positive externality game, as with $N_A = S_A^{.7}S_B^{.3} - cS_A^2$ for player A and $U_B = S_B^{.7}S_A^{.3} - cS_B^2$ for player B.

i. Differentiating N_A with respect to S_A allows A's best reply to be characterized and then solved for.

- The first order condition is $.7S_A^{-.3} S_B^{.3} - 2cS_A = 0$ at S_A^* (This is the familiar MB = MC condition.)
- To find A's best reply function solve for S_A^*. First shift the marginal cost function to the left.
- $.7S_A^{-.3} S_B^{.3} = 2cS_A$ Multiply both sides by $S_A^{.3}$, which yields:
- $.7S_B^{.3} = 2cS_A^{1.3}$ Divide both sides by 2c and then take the 1.3th root of both sides.
- $[.35S_B^{.3} /c]^{(1/1.3)} = S_A$ or, switching sides:
- $S_A^* = [.35S_B^{.3} /c]^{(1/1.3)}$ This is player A's best reply function. Note that his or her best replay varies with the choice of the other player, here S_B.)
- Player B's best reply function will look similar. (Deriving it is good practice.)
- At the Nash equilibrium both players will be on their best reply functions—which is to say that the Nash equilibrium occurs where the two best reply functions intersect one another.

ii. In symmetric games (games where the payoff functions (net benefits or utilities) are mirror images of one another, there is usually an equilibrium at which both players play the same strategies (as in many of the discrete strategy games above and in the previous chapter.)

- In that case, the symmetric equilibrium can be found by assuming that $S_A^* = S_B^*$
- This allows you to use the above reaction function to write:
- $S_B = [.35S_B^{.3} /c]^{(1/1.3)}$
- Solving this for SB will characterize both A's and B's strategies at the symmetric Nash equilibrium. To simplify the exponents a bit, raise each side to the 1.3 power:
- $S_B^{1.3} = [.35S_B^{.3} /c]$ Then divide both sides by $S_B^{.3}$, which yields:
- $S_B^{**} = [.35/c]$ This is player B's strategy at the Nash equilibrium
- Given our assumption of a symmetric equilibrium, Player A uses the same strategy so $S_A^{**} = [.35/c]$
-
- The analysis can be extended to address normative issues by contrasting the Nash equilibrium of this game with the strategy values that maximizes social net benefits, which in this case is simply the sum of the two net benefit functions. As on might suspect, both players under invest in their strategies.

iii. The process of finding Nash equilibria in games with an infinite number of strategies nearly always follows this general pattern.

III. Another Illustrating Example: Purchasing Lottery Tickets

A. Consider another **two-person contest, namely a lottery game** in which each can buy as many tickets as they like and each player's probability of winning depends on the number of tickets owned by that player relative to the total sold. Both players are rational and so want to maximize their "expected" net earnings from purchasing tickets.

i. The **expected value** of an event with outcomes 1, 2, i, ... N is $V^e = \Sigma$ PiVi, where Pi is the probability of event i, and Vi is the value of event i.

- If Al purchases Na lottery tickets and Bob purchase Nb tickets, Al's expected profit is $Ra^e = [Na / (Na + Nb)]Y - Na\ C$ where Y is the prize one and C is the cost of a lottery ticket.
- Similarly, Bob's expected net benefit (profit) is $Rb^e = [Nb / (Na + Nb)]Y - Nb\ C$

ii. Al's expected profit maximizing number of lottery tickets can be found by differentiating Ra^e with respect to Na and setting the result equal to zero.

- $dRa^e/dNa = \{[1 / (Na + Nb)] - [Na / (Na + Nb)^2]\}Y - C = 0$ at Na*
- Putting terms over the same denominator and adding C to each side yields:
- $[Na + Nb - Na]/(Na + Nb)^2 = C/Y$ or $Nb/(Na + Nb)^2 = C/Y$
- Next we want to solve for Na
- $Nb = (Na + Nb)^2\ C/Y$ or $Nb(Y/C) = (Na + Nb)^2$
- which implies that $(NbY/C)^{1/2} = Na + Nb$
- Thus, $Na^* = - Nb + (NbY/C)^{1/2}$

iii. This last function is sometimes called a **best reply function**. In this case, it tells Al the expected profit maximizing number of lottery tickets to purchase given any particular purchase by Bob.

- Note that Na* varies with Bob's purchase which implies that Al does **not** have a dominant strategy.
- Note also that a best reply function can be derived for Bob, $Nb^* = - Na + (NaY/C)^{1/2}$

iv. Note also that if both **persons are simultaneously on their best reply function**, neither can change their strategy and improve their payoff (remember that the best reply function for player i maximizes his or her payoff, given the strategies adopted by all other players), as required for the existence of a **Nash equilibrium**.

v. Thus, the **Nash equilibrium** of this lottery game occurs at a point where: $Na^* = - Nb^* + (Nb^*Y/C)^{1/2}$ and $Nb^* = - Na^* + (Na^*Y/C)^{1/2}$

- To find the Na* and Nb* combination where both these conditions hold, one can either substitute the equation describing Nb* in terms of Na into the Al's best reply function and do a bit of algebra.

vi. In a **symmetric game** (a game in which players have the same strategy sets and payoff functions) there is normally a symmetric equilibrium. In this case, the two best reply functions will intersect at a point where Na = Nb.

- At the symmetric lottery game's equilibrium: $Na = - Na + (NaY/C)^{1/2}$
  or $2Na = (NaY/C)^{1/2}$
- Squaring both sides, we have: $4Na^2 = NaY/C$ which implies that 4Na = Y/C
- or Na** = Y/4C and since Na = Nb at the symmetric Nash equilibrium, we also have Nb** = Y/4C
- Since each ticket costs C euros, so Al spends Na** C or Y/4 euros on tickets. That is he spend exactly 1/4 of the prize money (if he wins) on tickets.
- [The same is true for Bob, so it is clear that this particular lottery will not be a "money maker" for its organizers.]

B. The lottery game can be generalized to think about a wide variety of games in which one's odds of winning a contest depends upon how much time, energy, wealth, etc. one invests in the game.

C. **Common applications of lottery contests** include the political rent-seeking games, originally developed by Gordon Tullock (1980), legal battles in court, research and development contests by firms, warfare, car racing, grades on university exams, etc..

D. The lottery game and its various applications **can also be generalized** to take account of more than 2 players, and to include "technologies" where the exponents on investments are subject to increasing or decreasing returns.

E. It is surprisingly easy to **generalize** this game by, for example, including N players rather than two.

i. Let K represent the total investment of the N-1 players, then the expected payoff of a "typical" player is:

- $Ra^e = [Na / (Na + K)]Y - Na C$

ii. Differentiating with respect to Na yields:

- $dRa^e/dNa = \{[1 / (Na + K)] - [Na / (Na +K)^2]\}Y - C = 0$

iii. Solving for Na, as above, yields:

- $Na^* = - K + (KY/C)^{1/2}$

iv. This equation is the **best reply function of a typical player** in the present N person game.

v. To find the symmetric equilibrium, note that K = (N-1) Na, so:

- $Na^* = - (N-1) + [(N-1)Na\ Y/C]^{1/2}$

vi. Solving for Na*, yields:

- { $Na^{**} = [(N-1)/ N^2]\ (Y/C)$

vii. Note that when N = 2, as above, $Na^{**} = (1/4)\ (Y/C)$, as before.

viii. The **total expenditure** on "rent seeking" is NC times this amount, or (N-1)Y/N, and this expenditure approaches Y in the limit as N approaches infinity.

F. Different technologies for increasing one's chance of winning can also be taken into account by assuming changing our assumptions about investments in the game (Na) affect the probability of winning the prize. For example we can take account of economies and diseconomies of scale by changing from P =Na/(Na + K), to $P = Na^d/ (\Sigma Ni^d)$.

i. The payoff function for a typical player now becomes:

- $Ra^e = [Na^d/ (\Sigma Ni^d)]Y - Na\ C$

ii. Differentiating with respect to Na now yields:

- $dRa^e/dNa = \{[dNa^{d-1} / (\Sigma Ni^d)] - Na^d\ (dNa^{d-1})\ / (\Sigma Ni^d)^2 \}Y - C = 0$

iii. To find the symmetric equilibrium, note that Na = Ni for all i = 1, 2, ... N, so:

- $\{[dNa^{d-1} / (NNa)] - Na^d\ (dNa^{d-1})\ / (N^2Na^2) \}Y - C = 0$, or putting the numerators over a common denominator and collecting a few terms:
- $\{[dNNa^{2d-1} - dNa^{2d-1})]\ / (N^2Na^{2d}) \}Y - C = 0$, or
- $\{[d\ (N-1)Na^{2d-1})]\ / (N^2Na^{2d}) \}Y - C = 0$

iv. Solving for Na*, yields the individual's number of tickets (level of resources invested in the contest) at the symmetric Nash equilibrium:

- $Na^{**} = [(N-1)/ N^2]\ (dY/C)$

v. Note that when d=1 and N=2, as above, $Na^{**} = (1/4)\ (Y/C)$, as before.

- However, the **total expenditure** on "rent seeking" is again NC times this amount, or d(N-1)Y/N.
- Note that total expenditures **will now exceed Y**, whenever d> (N-1)/N.

IV. To summarize our analysis of lottery contests and contests that can be characterized as lotteries:

i. The more players are in the game, the less each spends.

ii. However, the total spent rises with the number of players.

iii. In games with constant returns (the classic contest function) the total investment in the contest approaches the value of the prize (Y) as the number of players approaches infinity.

iv. Contests with increasing returns may have "super dissipation," where more resources will be invested in the contest than the prize is worth.

v. (Note that no player will routinely play such games. However, "no one" playing is also not an equilibrium, so potential players may play mixed participation strategies--more on that later in the course.)

G. There are a surprisingly large number of applications of these rent-seeking-lottery games.

- Essentially any contest in which additional resources increases the probability of winning, or the fraction of the prize that is won, can be modeled with such functions.
- Indeed, a very large "contest" literature has emerged in the past ten or twenty years that explores such functions.
- To this point, the "Tullock" contest function has been most widely applied to represent interest group politics, although it can be used to represent crime, terrorism, etc. as noted above.
- Note that dissipation--the cost of the "competition"--is an important indicator of social welfare, particularly in contests that are "unproductive" and therefore wholly redistributive.

H. Game theory can also be used to represent less concrete settings.

- For example, payoff fuctions can be represented using abstract functions.
- And, equilibrium strategies can be characterized using a bit of calculus.

I. Illustration: consider a symmetric game in which each player has the same strategy set and the same payoff function.

i. Suppose there are just two players in the game, Al and Bob.

- Let the payoff of player A be G1 = g(X1, X2) and that of player B be G2 = g(X2, X1) where X1 is the strategy to be chosen by player 1 and X2 is the strategy chosen by player 2.

ii. Each player in a Nash game attempts to maximize his payoff, given the strategy chosen by the other.

- To find payoff maximizing strategy for player A, differentiate his payoff function with respect to X_1 and set the result equal to zero.
- The implicit function theorem implies that his or her best strategy X_1^* is a function of the strategies of the other player X_2, that is that $X_1^* = x_1(X_2)$.
- A similar reaction (or best reply) function can be found for the other player.

iii. At the Nash equilibrium, both reaction curves intersect, so that

$$X_1^{**} = x_1(X_2^{**}) \text{ and } X_2^{**} = x_2(X_1^{**})$$

V. A Few Practice Problems

A. Let R be the "reward from mutual cooperation," T be the "temptation of defecting from mutual cooperation," S be the "suckers payoff" if a cooperator is exploited by a defector, and P be the "Punishment from mutual defection." Show that in a two person game, relative payoffs of the ordinal ranking T > R > P > S are sufficient to generate a prisoner's dilemma with mutual defection as the Nash equilibrium.

B. Write down an assurance game and assume that the players initially find themselves at the less desirable Nash Equilibrium. Show that your trust problem can be solved by subsidies of various kind. Explain how this game differs from a PD game. Can subsidies also be used to solve a PD game?

C. Suppose that the inverse demand curve for a good is P = 100 - Q and that there are two producers. Acme has a total cost curve equal to C = 5Q and Apex has a total cost curve of C =10 Q. Each firm controls its own output. Prices are determined by their combined production. Characterize the Cournot-Nash equilibrium to this game.

D. Suppose that there are two neighbors, Ms 1 and Ms 2, each of whom enjoy playing their own music loudly enough to annoy the other. Each young woman maximizes a utility function defined over other consumption, C, here, the volume of their own noise (a good), and that of their neighbor's consumption, here, the volume of her neighbor's music (a bad). Ms 1's utility function is $U_1 = C_1^{0.5} N_1^{0.5} N_2^{-0.5}$. Ms 2 has a similar utility function and each has a budget constraint of the form , Yi = Ci + Ni.

i. Characterize each neighbor's "best reply" or "reaction" function, and then determine its slope.

ii. What happens to neighbor 1's reaction function if his income rises?

iii. Show the effect that a simultaneous increase in each neighbor's income has on the Nash equilibrium of this game.

iv. Is there anything strange about this game?

VI. Optional Appendix: A Short, but Abstract, Proof of the Existence of a Nash Equilibrium in Every Finite Player, Finite Strategy Game, Finite Repeated Game.

The proof of the existence of the existence of a Nash equilibrium for any finite game is short, but quite abstract. It relies on two ideas that we've not covered in this book: first a generalized form of continuity called upper hemi continuity which applies to functionals as well as functions, and second, Kakutani's fixed point theorem which is a generalization of Brower's fixed point theorem, extended to cover sets as well as lines. (Brower's fixed-point theorem basically says that any continuous function from a bounded line segment onto itself (as with a function on the interval 0-1 to the interval 0-1 will at some point reach a point where f(x) = x.

Such a point is called a "fixed point" because the function from x into y takes one back to the same point on the Y axis. The Kakutani fixed point theorem establishes a similar property for upper hemi-continuous functionals. (Functionals are mappings from one set into another set, whereas a function is a mapping from a set into a single point on a real number line).

Sufficient Conditions for the Existence of Nash Equilibria in Non-Cooperative Games

A. **Proposition.** Every finite player, finite strategy game has at least one Nash equilibrium if we admit mixed strategy equilibria as well as pure. (Kreps p.409 and/or Binmore p.320).

B. The proof relies upon Kakutani's fixed point theorem, which is a generalization of Browers fixed point theorem. Here is a condensed version. (Presented in Kreps, page 409).

i. Let i = 1, ... I be the index of players, let S_i be the (pure) strategy space for player i and let Σ_i be the space of probabilities distributions on S_i.

ii. The strategy space of mixed strategy profiles is $\Sigma = \Pi_{i=1}$ $\Sigma = \Pi_{i=1}^{I}\Sigma_i$, that is the cross product of all individual mixed strategies.

iii. For each combination of mixed strategies $\sigma = (\sigma_1, \sigma_2, ... \sigma_I)$ find each person's best reply function for player i, given the other strategies, $\Phi i\ (\sigma^{-i})$. (Here, σ^{-i}, denotes σ less i's dimension of σ)

iv. Define $\Phi = (\Phi_1, \Phi_2, ... \Phi_I)$ as the vector of best reply functions. *Note that* Φ *is a mapping from the domain of mixed strategies onto itself.*

v. It is upper semicontinuous and convex (*not proven here*), hence by Kakutani's fixed point theorem, a fixed point exists. (Intuitively, expected value functions are concave (linear) and continuous so Φ is also concave and continuous.)

vi. **This fixed point is a mixed strategy I-tuple which simultaneously characterizes and satisfies each player's best reply function**. This is it is a Nash equilibrium. Q. E. D.

- A somewhat less general, but more intuitive proof is provided by Binmore in section 7.7.