

## I. An Introduction to Mathematical Economics.

Scope of Course, Usefulness and limitations of deductive methodology. Basic Concepts: compact sets, convexity, continuity, functions, Application: preference orderings, fundamentals of consumer theory.

- A.** Essentially all scientific work attempts to determine what is general about the world.
- i. For example, successive sun rises may be more or less beautiful but all sun rises on Earth are caused by the Earth's daily rotation in combination with light generated by our nearest star..
  - ii. Persons take account of many different characteristics of an automobile when they select a car, but all at some point all must consider the price of the car. Economics argues that the higher is the price the less likely a given person is to purchase a particular car, other things being equal.
- B.** This process of finding relationships or general rules of thumb is complex but may itself be described as a joint exercise in logic (model building) and observation (empirical testing)--or perhaps more accurately as a rough cycle of model building, empirical testing, refinement, retesting, and so, forth.
- i. This course attempts to provide students with the core mathematical concepts and tools that are the most widely used by economists in the "model building" part of the scientific enterprise.
  - ii. Model building always starts with ideas and simplifications; however, once a model has been deemed useful, "standard" forms of models tend to emerge.
    - ◆ Most graphical models have mathematical foundations: that is to say, the math was usually worked out before the geometry.
    - ◆ Utility functions are mathematical functions that represent utility. Production functions are mathematical functions that represent output as functions of inputs.
    - ◆ Macroeconomic models often begin with an equation, as in the  $Y=MV$  equation of monetary economics, or a system of equations as in basic Keynesian models.
    - ◆ General equilibrium models and social welfare functions were all worked out mathematically first, and are generally clearer in their mathematical expression than in their geometric ones.
    - ◆ Game theoretic notions of best reply functions or reaction functions are clearly mathematical in origin, and usually clearer in the math than in their graphical representations.
    - ◆ This is not to say that graphs are a waste of time. In many cases our geometric intuition is stronger than our algebraic or calculus intuitions, and so we can "squeeze" more out of a geometric representation than from equations.
  - iii. It also bears noting that the empirical part of economics has mathematical foundations. Econometrics uses various statistical methods to "test" hypotheses generated from economic models.
    - ◆ Statistical tools nearly all have mathematical foundation.
    - ◆ Most statistical tests assume that you know the right functional form of the models that you estimate and of the relevant error distributions (normal, etc.).
    - ◆ Not all persons who use econometrics write down their models, but they implicitly are making mathematical assumptions, when they use linear representations for their estimates (as in regression analysis and generalized least squares).
- C.** Not all economic models are mathematical, but mathematical models have many advantages over other modeling methods.

- ◆ Two of the most important are LOGICAL CONSISTENCY (what you have derived is true, given your assumptions) and CLARITY (you know, or should know, what you have assumed).
- ◆ Another is that the mechanics of math allow you to push the logic of a model to places where your intuitions stop functioning or become confused.

**D.** Mathematical models allow mathematics (deduction) to be used as an "engine of analysis."

- i. If the model is "close enough" then properties deduced from the models will also be properties of the real world.
- ii. Those deductions, by dealing with simpler representations of the choice settings of interest (models) often have surprising and/or counter intuitive predictions that can be subjected to empirical tests.
- iii. [Moreover, it bears noting that most statistical tests assume that you already know the mathematical form of the model.]

**E.** Economists often attempt to deduce some consequences of purposeful decision making using "lean models," that is to say models that make very few assumptions about decision making in a setting of scarcity.

- i. Typical assumptions assume fundamental properties of individual purposes--which are represented with preference orderings or utility functions.
- ii. In addition, a choice setting is generally characterized with a feasible set of some kind--as with a budget set or budget line, technological production possibility set or function, or conditional probability function..
- iii. As will be seen in this course, the design of economic model is partly based on economic considerations, partly on mathematical ones, and partly on "professional conventions."

**F.** Methodological individual is an approach to social phenomena that assumes that all social phenomena--including markets--can be analyzed in terms of the individuals in the group or society of interest and the external constraints they confront (budgets, laws, technology, strategy sets, etc.).

- i. To model individuals, "we" normally assume that individuals have goals that can be characterized mathematically.
  - a. Rational individuals may be "net-benefit" maximizers.
    - ◆ Given  $B=b(Q)$  and  $C=c(Q)$ , we assume that "Al" chooses  $Q$  to maximize  $N = B-C$ .
  - b. Rational individuals may be "utility" maximizers.
    - ◆ Given  $U=u(a, b)$  where  $W=a+b$ , we model "Bob's choice as finding the combination of  $a$  and  $b$  that maximize  $U$  for given  $W$ .
  - c. Rational individuals may be people who simply make consistent choices.
    - ◆ Al prefers A to B and B to C, does Al prefer A to C? Why or why not?
- ii. All three of these models are widely used by economists and choices by such individuals can be represented mathematically (with sets and equations) as well as with diagrams.
  - a. The models are all very similar to one another in spirit.
  - b. However, there are subtle assumptions that one makes as one shifts from: consistent preference orderings to utility maximizing to net-benefit maximizing models.
  - c. The utility maximizing models are the most widely used of the three, but the others are also widely used, the former in mathematical economics and the latter in more applied work.
  - d. [In addition, there are extensions of these models to take account of specific circumstances.

- e. A person that owns a firm may be modeled as a profit maximizer.
- f. A person that is running for an elective office in a democracy may be modeled as a vote maximizer.
- g. In setting where risk or time are important, the persons may be modeled as maximizing “expected” utility or net benefits, or the present value of utility flows or net benefits.

**G.** Nearly all models that can be represented with equations can also be represented with diagrams.

**H.** This course will focus for the most part on mathematical approaches, especially those using basic tools from calculus.

- i. Calculus allows a lot of fairly general results to be worked out, without making too many outrageous assumptions.
- ii. Many of the results are much more general than can be shown with diagrams, and in some cases more intuitive (once you used to thinking mathematically about choices and choice settings).
  - ◆ It bears noting that most geometric representations were first worked out mathematically, and make sense because of their mathematical foundations.]
- iii. My goal in this course is to teach you how to model choice settings mathematically, and the subset of tools that I focus on have been chosen with that in mind.
- iv. The course is mostly calculus based, but as today we will do a bit of more abstract analysis, and also use diagrams to illustrate many of our results.
  - ◆ The class lectures are not simply me reading my lecture notes.
  - ◆ There will be more diagrams in my lecture than in the lecture notes.
  - ◆ The lectures are the best time to ask questions. (Nearly all questions are welcome!)

## II. A Point of Departure: the Fundamental Geometry of Net-Benefit Maximizing Choice

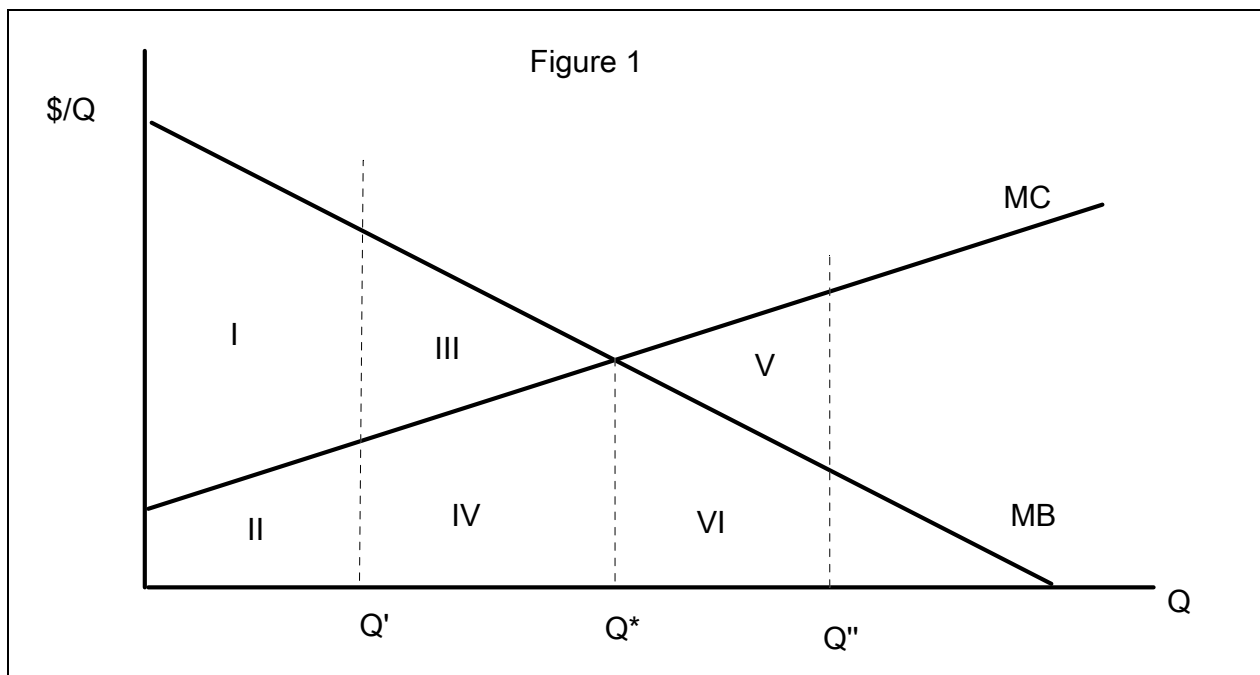
**A.** Nearly all economic models can be developed from a fairly simple model of rational decision making that assume that individuals **maximize** their private **net benefits**.

- i. Consumers maximize consumer surplus: the difference between what a thing is worth to them and what they have to pay for it.  $CS(Q) = TB(Q) - TC(Q)$
- ii. Firms maximize their profit; the difference in what they receive in revenue from selling a product and its cost of production:  $\Pi = TR(Q) - TC(Q)$

**B.** The change in benefits, costs, etc. with respect to quantity consumed or produced is generally called Marginal benefit, or Marginal cost.

- i. DEF: **Marginal "X"** is the change in Total "X" caused by a one unit change in quantity. It is the slope of the Total "X" curve.  $X \in \{\text{cost, benefit, profit, product, utility, revenue, etc.}\}$
- ii. *Important Geometric Property:* Total "X" can be calculated from a Marginal "X" curve by finding the area under the Marginal "X" curve over the range of interest (often from 0 to some quantity Q). This

property allows us to determine consumer surplus and/or profit from a diagram of marginal cost and marginal revenue curves.



### C. Examples:

- i. Given the marginal cost and marginal benefit curves in Figure 1, it is possible to calculate the total cost of  $Q'$  and the total benefit of  $Q'$ .
    - a. These can be represented geometrically as areas under the curves of interest.  $TC(Q') = II$  ;  $TB(Q') = I + II$  .
    - b. Note that calculating these areas is equivalent to finding the definite integral of the MC or MB functions over the range of interest (about which we will say more later in the course).
  - ii. One **can calculate the net benefits** by finding total benefit and total cost for the quantity or activity level of interest, and subtracting them.
    - a. Thus the net benefit of output  $Q'$  is  $TB(Q') - TC(Q') = [I + II] - [II] = I$ .
    - b. This is equivalent to finding the integral of  $MB(Q) - MC(Q)$  for the range of interest, here from 0 to  $Q'$ .
  - iii. Use Figure 1 to determine the areas that correspond to the total benefit, cost and net benefit at output  $Q^*$  and  $Q''$ .
  - iv. Answers:
    - a.  $TB(Q^*) = I + II + III + IV$  ,  $TC(Q^*) = II + IV$  ,  $NB(Q^*) = I + III$
    - b.  $TB(Q'') = I + II + III + IV + VI$  ,  $TC(Q'') = II + IV + V + VI$  ,  $NB(Q'') = I + III - V$
- D.** If one attempts to maximize net benefits, it turns out that generally he or she will want to consume or produce at the point where marginal cost equals marginal benefit (at least in cases where  $Q$  is very divisible).
- i. There is a nice geometric proof of this principle from calculus.
  - ii. In the example above, C, nearly proves this. Note that  $NB(Q^*) > NB(Q')$  and  $NB(Q^*) > NB(Q'')$ .

- iii. In the usual chase, a net-benefit maximizing decision maker chooses consumption levels ( $Q$ ) such that their own marginal costs equal their own marginal benefits. They do this not because they care about "margins" but because **this is how they maximizes net benefits** in most common choice settings of interest to economists.
  - iv. This characterization of net benefit maximizing decisions is quite general, and can be used to model the behavior of both firms and consumers in a wide range of circumstances.
  - v. Moreover, the same geometry can be used to characterize ideal policies if "all" relevant costs and benefits can be computed, and one wants to maximize *Social Net Benefits*.
- E.** That each person maximizes their own net benefits does not imply that every person will agree about what the ideal level or output of a particular good or service might be.
- i. Most individuals will have different marginal benefit or marginal cost curves, and so will differ about ideal service levels.
  - ii. To the extent that these differences can be predicted, they can be used to model both private and political behavior:
    - a. (What types of persons will be most likely to lobby for subsidies for higher education?)
    - b. What types of persons will prefer progressive taxation to regressive taxation?
    - c. What industries will prefer a carbon tax to a corporate income tax?)
  - iii. Note, however, that one's best choice does not always equate MB and MC. For example, a very common choice for most consumers is to choose  $Q^* = 0$ . (How many pink cadillacs do you own?) Show that MB doesn't "usually" equal MC in this case.
- F.** One can **use the consumer-surplus maximizing model to derive a consumer's demand curve** for any good or service (given their marginal benefit curves) by: (i) choosing a price, (ii) finding the implied marginal cost curve for a consumer, (iii) use MC and MB to find the CS maximizing quantity of the good or service, (iv) plot the price and the CS maximizing  $Q^*$ , and (v) repeat with other prices to trace out the individual's demand curve.
- G.** Similarly, one can use a profit maximizing model (another measure of net benefit) to derive a competitive firm's **short run** supply curve, given its marginal cost curve. Again, one (i) chooses a price (which is a price taking firm's MR curve), (ii) finds the profit maximizing output, (iii) plot P and  $Q^*$ , (iv) repeat to trace out a supply curve.

### III. Axiomatic Choice: Some Fundamental Concepts and Definitions from Mathematics

- A.** We now shift from one of models with the "strongest" assumptions to some models of choice that make relatively "weaker" assumptions about preferences.
- B.** When the rules of logic are applied to numbers the result is mathematics.
- i. Most of the mathematics we have been taught can be deduced from a few fundamental assumptions using the laws of logic.
    - ◆ (See the postulates of Peano, an Italian mathematician (1850 - 1932).)
  - ii. Some mathematical economists are very attracted to the axiomatic approach (See Debreu's book for an early example), and we will spend a bit of time today seeing how that approach can be used to model human choices.
- C.** Some fundamental properties of logical and mathematical **relationships**:
- i. DEF: Relationship R is **reflexive** in set X, if and only if  $aRa$  whenever a is an element of X.

- ii. DEF: Relationship R is **symmetric** in set X if and only if  $aRb$  then  $bRa$  whenever a and b are elements of set X.
  - iii. DEF: Relationship R is **transitive** in set X if and only if  $aRb$  and  $bRc$  then  $aRc$  when a, b, and c are elements of set X.
  - iv. Recall that within the set of real numbers, there are relationships which are symmetric (equality), reflexive (equality) and transitive (equality, greater than, less than, greater than or equal than, less than or equal than).
  - v. In economics there are also several relationships which possess all three properties, and some that exhibit only transitivity.
    - a. In general, economists assume that “rational” preference orderings satisfy all three of these properties. Strong and weak preference orderings are transitive, while indifference is transitive, symmetric and reflexive.
    - b. Indeed, *rationality in microeconomics is often defined as transitive preferences.*
  - vi. Note that the indifference relationship, I, can be defined in terms of the weak preference relationship R.
    - a. The weak preference relationship R means "at least as good as."
    - b. Note that if  $aRc$  and  $cRa$ , then  $aIc$ .
  - vii. Similarly, the strong preference relationship, P, can be defined in terms of the weak preference relationship.
    - a. The strong preference relationship means "better than."
    - b. Note that if  $aRb$  but  $b \sim Ra$  then  $aPb$ .
- D.** DEF: A **function** from set X to (or into) set Y is a rule which assigns to each x in X a unique element,  $f(x)$ , in Y. Set X is called the domain of function f and set Y its range.
- ◆ On most diagrams from math classes, the domain is the horizontal axis (X) and the range is the vertical axis (Y).
  - ◆ However, most textbook diagram in economics have the domain of the function on the vertical axis (P) and the range of the function on the horizontal axis (Q). For example demand functions go from P (prices) into Q (quantities a consumer is prepared to purchase).
  - ◆ This is evidently Marshall’s fault, who decided that the diagrams were easier to draw this way.
- E.** DEF: A **utility function** is a function from set X into the real numbers such that iff  $aPb$  then  $U(a) > U(b)$  and if  $aIc$  then  $U(a)=U(c)$  for elements of the set X.
- ◆ Note that a utility function defined in this way does not characterize satisfaction, but rather choices based on transitive weak preferences.
  - ◆ However, if you think of utility as “satisfaction” or closeness to some “goal” you intuition will be correct.
  - ◆ In diagrams, indifference curves are simply graphs (plots) of all combinations of two goods that generate the same utility.
- F.** Note that the assumption that a utility function exists, is equivalent to the assumption that individual preferences are such that: preferences are transitive;
- i. Real numbers are transitive with respect to equality (indifference) and greater than (strict preference).
  - ii. The assumption that all combinations of the goods can be evaluated with the function implies that preferences are **complete**: each bundle (combination of goods and/or "bads") has a unique rank.
    - a. Every bundle of goods generates either more or less or the same utility level as other goods.

- b. (Some theorists make a distinction between complete and incomplete utility mappings from  $X$  to  $R$ , but this distinction is not important for "routine" decisions.. Why?)

**G.** Some important definitions and concepts from Set Theory.

- i. DEF: An infinite series,  $x_1, x_2, \dots, x_n$  is said to have a **limit** at  $x^*$  whenever for any  $d > 0$ , the interval  $x^* - d, x^* + d$  contains an infinite number of points from the series. (That is to say,  $x^*$  is a limit point of a series in any case where there are an infinite number of elements of the series arbitrarily close to  $x^*$ .)
- ii. DEF: A set is **closed** if it contains all of its limit points.
- iii. Def: A set is **bounded** if every point in  $A$  is less than some finite distance,  $D$ , from other elements of  $A$ .
- iv. Def: A set is **compact** if it is closed and bounded.
  - ◆ Most opportunity sets in economics are assumed to be closed and bounded.
- v. Def: A set is **convex** if for any elements  $X_1$  and  $X_2$  contained in the set, the point described as  $(1-\alpha)X_1 + \alpha X_2$  is also a member of the set, where  $0 < \alpha < 1$ .
  - a. Essentially a convex set includes all the points directly between any two points in the set.
    - ◆ That is to say, any convex (linear) combination of two points from the set will also be a point in the set.
    - ◆ Thus a solid circle, sphere, or square shaped set is a convex set but not a V-shaped or U-shaped set. What other common geometric forms are convex?
    - ◆ To see this, draw one of these figures, pick some representative pairs of points and connect them with a line interval (cord). The line interval will lie entirely in the set.
    - ◆ (Why does a series of convex combinations trace out a cord? Because as  $\alpha$  varies from 0 to 1, the "point" characterizes movement along a line from one point ( $X_1$ ) to the other ( $X_2$ ).
  - b. Example from economics: usually "better sets" are assumed to be convex sets.
    - ◆ That is to say, the set of all bundles which are deemed better than bundle  $a$  is generally assumed to be a convex set.
    - ◆ Another example, is the budget set, the set of all affordable commodities given a fixed wealth and fixed prices for all goods that might be purchased.
- vi. **Convexity and compactness assumptions are widely used calculus and graphical models** of human decisionmaking, because they make smooth continuous functions (and lines) possible.
  - ◆ Opportunity sets and production possibility sets are nearly always assumed to be convex and compact.

**H.** Some important concepts and definitions from and for **Calculus**

- i. Def: Function  $Y = f(X)$  is said to be **continuous** whenever the limit of  $f(X)$  approaches  $Y = f(Z)$  as  $X$  approaches  $Z$ .
  - a. Or alternatively, function  $Y = f(X)$  is said to be continuous if for every point in the domain of  $X$ , and for any  $\epsilon > 0$ , there exists  $\delta > 0$ , such that  $|f(X) - f(Z)| < \epsilon$  for all  $X$  satisfying  $|Z - X| < \delta$ .
  - b. (That is to say, points only a finite distance from  $Z$  should generate function values within a finite distance of  $f(Z)$ .
    - ◆ In fact,  $f$  is continuous if for any finite distance  $\epsilon$  (epsilon) there exists  $\delta$  (delta) such that any value within  $\delta$  of  $z$  generates a function value within  $\epsilon$  of  $f(z)$ .
    - ◆ This condition assures that there are no "sudden" (instantaneous) jumps in the function and no holes in the function.

- ◆ [Note that  $Y = 1/X$  is not continuous at 0, because as one gets close to Zero  $Y$  increases by more and more, some of these increases will exceed the “ $d$ ” chosen.]
  - ii. Def: the limit of a function: function  $f$  is said to have a **limit point**  $y^*$  at  $x^*$  if and only if (iff) for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $|f(x) - y^*| < \epsilon$  whenever  $|x - x^*| < \delta$ .
    - a. The limit of  $f$  at  $x^*$  is denoted  $\lim_{x \rightarrow x^*} f(x) = y^*$
    - b. If there is a real number  $y^*$  satisfying this definition at  $x^*$ , we say that the limit of  $f$  at  $x^*$  exists.
    - c. Note that this definition rules out the existence of different right hand and left hand limits. (why?)
  - iii. Def: function  $f$  is said to be **differentiable** if and only if (iff) for every  $x$  contained in set  $X$  the limit point of  $\{ [(f(x) - f(z))/(x - z)] \}$  exists.
    - ◆ Note that if  $f$  is differentiable,  $f$  is also continuous. (why?)
- I. Within microeconomics, utility functions and production functions are generally assumed to be continuous and twice differentiable.**
- i. Such assumptions clearly rule out some kinds of *decision makers*, just as the assumption that production possibility sets and opportunity sets are convex and compact rule out some kinds of *choice settings*.
  - ii. These assumptions are made largely for "economic" rather than "empirical" reasons.
    - ◆ That is to say, generally it is felt that the benefits of more tractable models overwhelms the costs of reduced realism and narrower applicability.
  - iii. However, this assumption should not always be taken for granted.
    - ◆ There are a few cases in which continuous versions of the choice settings lead to empirically false predictions.
    - ◆ Clearly, such uncommon choice settings should not be entirely neglected, although the continuity assumption will have to be dropped to analyze such cases or the models modified in other ways.
    - ◆ When discrete aspects of the choice problem are, or may be, important, various tools from set theory, integer programming, and real analysis can still be applied.
  - a. However, for most choice settings of interest to economists, the assumption of continuity is approximately correct. There may be a smallest piece of flour, sugar, gasoline, or sand, but they are pretty small!
    - ◆ Try to think of cases where the simplifying assumptions of continuity and convexity will generate predictions about behavior that are clearly wrong.

#### IV. Problem Set (Collected next week)

- A.** Suppose that Al always prefers larger apartments to smaller ones, but is unable to discern difference of ten sq. ft. or less. Are Al's preferences transitive?
- ◆ Explain why or why not.
- B.** Determine whether the following sets are convex sets or not.
- i. Al has a budget set  $W > P_a A + P_b B$ , where  $A$  and  $B$  are both non negative numbers.  $P_a$  is the price of good  $A$  and  $P_b$  is the price of good  $B$ .  $W$  is Al's budget. Is Al's budget set convex?
  - ii. Barbara has a bliss point "B" characterized in terms of all goods relevant over which her preferences are defined. Consider bundle "C" which has less of every good than B.
    - ◆ Assume a set of indifference curves. Is the better set for "C" convex?
    - ◆ Repeat with another set of indifference curves.



- ◆ Are better sets always convex? (If not, draw a counter example and explain.)

**C.** Consider the function  $f(X) = 1/X^2$  for  $X \neq 0$  with  $f(X) = 1$  for  $X = 0$ .

- i. Is the domain of  $f$  compact?
- ii. Is  $f$  a continuous function?
- iii. Is  $f$  monotone increasing for  $x > 0$ ?
- iv. Prove that the  $f$  does not have a limit at  $X=0$ .

**D.** For most purposes, economists assume that utility functions are continuous and twice differentiable.

- i. What does this differentiability imply about the commodity space over which the utility function is defined?
- ii. What does twice differentiability imply about the shape of the utility function?
  - ◆ Hint: think about what a finite first and second derivative mean. For example, the first derivative of a utility function is marginal utility. The second derivative of a utility function is the slope of the marginal utility curve. What does a finite and continuous second derivative imply about the slope of a marginal utility function?
- iii. Are there any important limitations of models that rely upon the assumption of differentiable utility functions? Discuss.
- iv. Is differentiability an important modeling assumption or a mathematical convenience? Explain.

*Next week: optimization, concavity, and rational choice*