

## An Introduction to Optimization

The core of modern economics is the notion that individuals optimize. That is to say, individuals use the resources available to them to advance their own personal objectives as well as they know how. In consumer theory, the consumer's decision is represented as an effort to maximize utility given a budget set. In the theory of the firm, firms are assumed to maximize profit, given production technology and market opportunities. In public choice theory, candidates are assumed to maximize votes given the positions of other candidates and the preferences of voters.

Such choices can all be characterized with the mathematics of constrained optimization in settings where the individual controls something that may be more or less continuously varied: time, money, policy position, output. In each case there is an objective (utility, profit, votes ...) and each case there are constraints which characterize a feasible set (budget set/line, production function, distribution of voter preferences ...).

Today's lecture develops necessary and sufficient conditions for the existence of a maximum and minimum (with and without constraints) and the use of derivatives in demonstrating the concavity of a function.

### I. More Fundamental Concepts and Definitions from Mathematics

- A.** Economic analysis usually models both individuals and firms as maximizers. Individuals maximize utility or net benefits. Firms maximize economic profit.
- B.** It turns out that in most cases there are theoretical and/or empirical reasons to believe that both utility functions and profit functions are concave.
- C.** Many of the mathematical properties of a given function can be deduced from its "shape." One of the most widely used notions of shape is *concavity*.
- i. Here are **three notions** of concavity:
  - ii. DEF: Strictly Concave: function  $f$  is *strictly* concave iff
 
$$\alpha f(X_1) + (1-\alpha)f(X_2) < f(\alpha X_1 + (1-\alpha)X_2) \quad \text{where } 0 < \alpha < 1.$$
 (That is, iff all the points on a cord connecting two points on a function lie beneath the function.)
  - iii. DEF: Concavity: function  $f$  is concave iff
 
$$\alpha f(X_1) + (1-\alpha)f(X_2) \leq f(\alpha X_1 + (1-\alpha)X_2) \quad \text{where } 0 \leq \alpha \leq 1.$$
 (Points on a cord connecting two points of a function lie beneath or on the function.)
  - iv. DEF: Quasi-Concave: Concavity: function  $f$  is *quasi* concave iff
 
$$f(X_1) \leq f(\alpha X_1 + (1-\alpha)X_2) \quad \text{where: } 0 \leq \alpha \leq 1$$
 and  $f(X_1) < f(X_2)$ .  
 (The function lies above, or not below, the lower of the two end points of a cord connecting points on the function.)
  - v. Any strictly concave function is also a concave function, but not vice versa.

- vi. Any concave function is also a quasi concave function, but not vice versa.
- vii.

**D.** Concave functions have a number of useful properties in the context of "maximizing" behavior.

- i. A *strictly* concave function has at most one maximum. (Draw some pictures to see why.)
- ii. However, a **concave function** may have an infinite number of global maxima, but if there is more than one maximum, they make up a continuous interval. (A horizontal line is concave.)
- iii. DEF: The **global maximum** of a function,  $f(x)$ , has a value which exceeds all others over the entire range of the function ( e. g. for every neighborhood of  $x^*$ ).
- iv. DEF: The **local maximum** of a function,  $f(x)$ , has a value which exceeds those of other points within a finite neighborhood of  $x^*$ . That is,  $f(x^*)$  is a local maximum if  $f(x^*+e) < f(x^*)$  and  $f(x^*-e) < f(x^*)$  for  $0 < e < E$ , for some  $E > 0$ .

**E.** Derivatives of functions can be used to characterize *sufficient* conditions for concavity, strict concavity, and therefore also for global maxima and minima.

- i. A function is *strictly concave* if its second derivative is negative over its entire domain.
- ii. A *strictly* concave function has at most one global maximum. Thus, if function  $f$  is strictly concave and  $f'(x^*) = 0$ , then  $f(x^*)$  is the global maximum of  $f$ .
- iii. A function is *concave* if its second derivative is less than or equal to zero over its entire domain.
- iv. A differentiable function is at a **local or global maximum or minimum** whenever its first derivative has the value zero.
- v. Given that one is at an extremal (an  $f(x)$  where the first derivative,  $f'(x)$ , is zero),
  - a. the extremal is a local maximum if the second derivative is negative,
  - b. the extremal is a local minimum if the second derivative is positive.
- vi. For multidimensional functions, a matrix of partial derivatives, called a *Hessian*, can be used to determine whether a multidimensional function is concave. At a local maximum, the Hessian will be negative definite. (See the Matrix Magic Handout.)
- vii. However, usually economists just assume that the relevant objective functions (utility, profit, net benefits, etc.) are concave or strictly concave.

**F.** Some Useful Derivative Formulas

- i.  $y = ax + C$   $dy/dx = a$
- ii.  $y = ax^b + C$   $dy/dx = bax^{b-1}$
- iii.  $q = ax^b y^c$   $dq/dx = abx^{b-1} y^c$
- iv.  $u = \log(x)$   $du/dq = 1/x$
- v.  $h = y(x) z(x)$  product rule  $dh/dx = (dy/dx)(z) + (y)(dz/dx)$
- vi.  $h = y(z(x))$  composite function rule  $dh/dx = (dy/dz(x))(dz/dx)$

**G.** To increase the generality of their models, economists generally use very abstract functions that are characterized only by partial derivatives and/or monotonicity and concavity--that is to say by their general shape.

- i. Economists normally *assume* that the functions of interest (utility functions, profit functions, production functions) are twice differentiable, concave, and monotone increasing over the range of interest.
- ii. The assumption that "more is better" is implicitly an assumption that the relevant functions are monotone increasing and, if differentiable, have positive first derivatives.

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- iii. The assumption that second derivatives are negative implies strict concavity and implies that the functions of interest exhibit diminishing marginal returns. (That is to say, the slopes of the marginal utility, marginal product, marginal benefit, etc., curves slope downward.)
  - iv. Occasionally, economists will analyze settings where the functions are not concave or strictly concave, as in cases in which production exhibits economies of scale.
    - (Cost functions are not usually assumed to be concave, except when constant returns to scale are assumed.)
- H.** For the purposes of illustration, or in order to mathematically derive a specific functional form for an economic relationship of interest ( a demand curve, cost function, etc.), “explicit” functional forms are often assumed for utility and production functions.
- i. For example, a utility function might be assumed to have an exponential or *Cobb-Douglas* form,  $U = ax^b y^c$ , where:  $x$  and  $y$  are quantities of two goods. In a Cobb-Douglas function the exponents sum to one,  $b + c = 1$ .
  - ii. At other times somewhat less restrictive assumptions are used. For example a production function might be assumed to be **homogeneous** of degree 1.
    - a. DEF. A function is said to be *homogeneous of degree  $k$* , if and only if whenever  $Y = f(X)$ , then  $f(\lambda X) = \lambda^k Y$
    - b. A production function that is homogenous of degree 1 exhibits constant returns to scale. Doubling all inputs doubles outputs.
    - c. Cobb-Douglas functions and linear functions through the origin ( $Y = ax$ ) are homogeneous of degree 1.
  - iii. Occasionally, utility and production functions are assumed to be *homothetic*, a somewhat more general family of functions than homogeneous functions.
    - a. DEF. A **homothetic function** is a composite function of the form  $H = h(Q(a,b))$  where  $Q$  is a homogeneous function and  $dH/dQ \neq 0$ .
    - b. All homogenous functions are homothetic functions but not all homothetic functions are homogeneous.
    - c. Homothetic utility functions have linear income expansion paths. Similarly, homothetic production functions have linear output expansion paths. (*The slopes of the isoquants are the same along any straight line through the origin.*)
  - iv. These assumptions make the results less general than they are with assumptions of monotonicity, concavity, and strict concavity, but the clarity of the results is often felt to warrant such assumptions.
    - a. There are also cases in which specific assumptions about functional forms are adopted before conducting statistical analysis.
    - b. For example, Cobb-Douglas production functions can be estimated using ordinary least squares (regression analysis) using the logs of the values in the data set. (Note that the log of a Cobb-Douglas function is linear.)

## II. Unconstrained and Constrained Optimization

- A.** Many optimization problems of interest to economists can be regarded as “unconstrained,” because there are no bounds placed on the control variables determined by the decision maker of interest.
- i. Firms can pick any  $Q$  they want to when they choose a production level to maximize profits.
  - ii. Consumers can pick any  $Q$  they want to maximize net benefits (consumer surplus).

- iii. Such choices can be easily modeled using calculus.
  - iv. The firm's choice can be regarded as that which sets marginal profits equal to zero, which requires marginal revenue to equal marginal cost, **because** maximizing  $\Pi = R(Q) - C(Q)$  requires setting  $d\Pi / dQ = 0$ , which requires  $dR/dQ - dC/dQ = 0$ .
  - v. Similarly, a consumer that maximizes consumer surplus or net benefits requires or marginal benefit to equal marginal cost, **because** maximizing  $N = B(Q) - C(Q)$  requires setting  $dN / dQ = 0$ , which requires  $dR/dQ - dC/dQ = 0$ .
  - vi. (Assuming that the objective functions  $\Pi$  and  $N$  are strictly concave and differentiable.)
- B.** However, many optimization problems facing individuals and firms involve constraints of one kind or another.
- i. For example, utility maximizing individuals are constrained by a "binding" budget constraint. When the consumer is choosing how much of goods  $X_1$  and  $X_2$  to purchase, his or her choice of  $X_1$  affects how much  $X_2$  can be purchased.
  - ii. In such cases, the above "unconstrained" maximizing (or minimizing) technique *cannot* be directly applied -- by either the individuals themselves or those attempting to model their behavior.
  - iii. However, it is often possible to "substitute" the constraint (s) into the objective function in a manner that allows the above maximization technique to be used even when there are constraints.
  - iv. Alternatively, one can use the Lagrangian technique (which is developed in the next lecture).
- C. The Substitution Method.** In many cases, it is possible to "substitute" the constraint(s) into the objective function (the function being maximized) to create a new composite function which fully incorporates the effect of the constraint.
- i. For example consider the separable utility function:  $U = x^5 + y^5$  to be maximized subject to the budget constraint  $100 = 10x + 5y$ . (Good  $x$  costs 10 \$/unit and good  $y$  costs 5 \$/unit. The consumer has 100 dollars to spend.)
  - ii. Notice that, from the constraint we can write  $y$  as:
 
$$y = [100 - 10x]/5 = 20 - 2x$$
  - iii. Substituting into the objective function (in this case, a utility function) yields a new function entirely in terms of  $x$ :  $U = x^5 + (20 - 2x)^5$ 
    - a. This function accounts for the fact that every time one purchases a unit of  $x$  one has to reduce his consumption of  $y$ .
    - b. So the new function includes the entire effect of the constraint.
  - iv. Differentiating with respect to  $x$  allows the utility maximizing quantity of  $x$  to be characterized as:
 
$$d[x^5 + (20 - 2x)^5] / dx = .5 x^{-.5} + .5(20-2x)^{-.5} (-2)$$
    - a. This derivative will have the value zero at the constrained utility maximum.
    - b. Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for  $x$  yields:
 
$$4x = 20 - 2x \Rightarrow 6x = 20 \Rightarrow x^* = 3.33$$
    - c. Substituting  $x^*$  back into the budget constraint yields a value for  $y^*$ 

$$y = 20 - 2(3.33) \Rightarrow y^* = 13.33$$
  - v. No other point on the budget constraint can generate higher utility than that at  $(x^*, y^*) = (3.33, 13.33)$ .

**III. Problem Set (Collected next week)**

- A.** Consider the demand function  $Q = a + bP + cY$ , with  $b < 0$  and  $c > 0$ .
- Find the slope of this demand function in the  $Q \times P$  plane.
  - Find the slope of this demand function in the  $Q \times Y$  plane.
  - Show that this demand function is homogeneous in prices iff:  $a = -cY$ .
  - Is the associated revenue function ( $R=PQ$ ) concave? strictly concave? (Hint: use the inverse demand function to characterize the price at which the firm can sell its output.)
  - What is the revenue maximizing quantity of this good?
  - Prove that this demand function is continuous in  $P$ .
- B.** Use the substitution method to:
- find the utility maximizing level of goods  $g$  and  $h$  in the case where  $U = g^a h^b$  and  $10 = g + h$ ,
  - find the utility maximizing bundle of goods when  $25 = g + h$  (i.e. if the wealth constraint is twice as high), and to
  - characterize the profit maximizing output of a firm where  $\Pi = pQ - C$  and  $c=c(Q, w)$ .

**Next week:** more applications of constrained optimization and an introduction to the Lagrangian method, with an appendix on sufficiency theorems and the Kuhn-Tucker technique.