

## Essentials of Constrained Optimization

Whenever a person uses scarce resources to advance his or her goals, the decision can be represented as an exercise in constrained optimization. A person at the grocery store tries to find the shopping basket full of food that best satisfies his or her appetite and health. A busy person allocates time between paid and unpaid work, travel, study, and recreation in order to realize the best life that he or she can imagine, given the time and opportunities at hand. A firm allocates its R&D budget among alternative projects to maximize expected profit.

Because so many business, engineering, and political decisions are constrained optimization problems, a good deal of mathematical research has been applied to developing alternative mathematical methods for solving these problems. As a result, there is both a vocabulary of optimization and an array of optimizing techniques.

The language of optimization distinguishes between **objective functions** (the goals: profit, utility, etc.), **constraints** (which describes what is feasible), control variables (the variables that can be directly determined by the decision maker) and parameters (variables that can not be controlled by the decision maker, or at least are taken as given for the present decision). [ DEF: the **feasible set** is the collection points that satisfies the constraints of the problem of interest. ]

There are **three mathematical optimization methods** that are widely used by economists: the substitution method, the Lagrangian method, and the Kuhn Tucker Method. Today's lecture will review these methods and discuss the conditions that are sufficient to fully characterize the optimal allocation of scarce resources among alternative uses.

### I. The Substitution Method

- A. In some cases, it is possible to "substitute" the constraint(s) into the objective function to create a new "composite function" that fully represents the effects of the constraint on the original objective function.
- i. Generally, the substitution method attempts to reduce the number of control variables that have to be taken account of.
  - ii. The substitution method generally entails the following steps:
    - a. First., use the constraints to completely specify of the control variables as functions of the subset of control variables that are of most interest.
    - b. Second, substitute these relationships into the objective function to form a new objective function (that reflects all the constraints).
    - c. Third, differentiate with respect to each of the control variables that remain. (Often this will be just a single variable.)
    - d. If the new objective function is strictly concave, the optimal value(s) of the control variable(s) given the constraint(s) is the one(s) that sets the first derivative equal to zero.
  - iii. For example consider the separable utility function:  $U = x^5 + y^5$  to be maximized subject to the budget constraint  $100 = 10x + 5y$ . (Good x costs 10 \$/unit and good y costs 5 \$/unit. The consumer has 100 dollars to spend.)
  - iv. We can rewrite the constraint as  $y = [100 - 10x]/5 = 20 - 2x$
  - v. Substituting this for Y in the objective function (here, the utility function) yields a new function *entirely in terms of x*:  $U = x^5 + (20 - 2x)^5$
  - vi. This new function accounts for the fact that every time one purchases a unit of x one has to reduce his consumption of y. (Why?)
  - vii. Note also that this new objective function has just one control variable, x.

viii. Differentiating with respect to  $x$  and setting the result equal to zero allows the utility maximizing quantity of  $x$  to be characterized:

$$d[x^{-5} + (20 - 2x)^{-5}] / dx = .5 x^{-6} + .5(20-2x)^{-6}(-2) = 0 \quad (\text{at } U \text{ max})$$

a. The derivative will have the value zero at the constrained utility maximum. Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for  $x$  yields:

$$4x = 20 - 2x \Rightarrow 6x = 20 \Rightarrow x^* = 3.33$$

b. Substituting  $x^*$  back into the budget constraint yields a value for  $y^*$

$$y = 20 - 2(3.33) \Rightarrow y^* = 13.33$$

ix. No other point on the budget constraint can generate a higher utility level than that  $(x^*, y^*) = (3.33, 13.33)$ .

### B. More Illustrations of the substitution method.

i. Derivation of Demand Curve, general case for two-good world

a. Suppose that Al's utility function is  $U = u(A, B)$ , with positive first derivatives, negative second derivatives, and positive or zero second derivatives.

b. Suppose also that she has  $W$  dollars to spend on these goods, whose prices are  $P^A$  and  $P^B$ .

c. The latter implies that Al's budget constraint is  $W = AP^A + BP^B$ .

d. Note that as long as "more as better" (positive first derivatives), she will choose a combination of goods along her budget constraint, rather than inside it. Note also that if she purchases  $A$  units of the second good, she will necessarily purchase  $B = [W - AP^A] / P^B$  units of the second good.

e. This relationship can be substituted into Al's utility function to create a "new" objective function that takes account of the budget constraint:  $U = u(A, [W - AP^A] / P^B)$

f. Differentiating this function with respect to  $A$  and setting the result equal to zero yields a first order condition that describes Al's purchase of good  $A$ .

g.  $U_A - U_B (P^A / P^B) = 0$  where  $U_A$  denotes the partial derivative of  $U$  with respect to  $A$  and  $U_B$  denotes the partial derivative with respect to  $B$ .

h. Note that this first order condition has a clear meaning. The first term,  $U_A$ , is the marginal utility of good  $A$ . The second term,  $- U_B (P^A / P^B)$ , is the (subjective) marginal opportunity cost of consuming  $A$  in terms of lost utility from good  $B$ .

i.  $U_A - U_B (P^A / P^B) = 0$  implies that at  $A^*$ ,  $U_A = U_B (P^A / P^B)$ , which means that  $A$  will be consumed at the level where marginal benefit (marginal utility) equals marginal cost (marginal opportunity cost).

j. The implicit function theorem--about which we will have more to say next lecture--allows  $A^*$  to be characterized as a function of the other variables in the first order condition:  $A^* = a(P^A, P^B, W)$ .

k.  $A^* = a(P^A, P^B, W)$  is Al's demand function for good  $A$ .

l. Al's demand function for good  $B$  can be solved by substituting this function into  $B = [W - AP^A] / P^B$ , so  $B^* = [W - A^*P^A] / P^B$ .

ii. A monopolist's profit maximizing output level can be characterized in a similar manner.

a. Suppose that the inverse demand function facing a monopolist is  $P = d(Q, Y)$ , with inverse demand (price) falling with increases in  $Q$  and increasing with increases in  $Y$  (consumer income).

b. (An inverse demand function maps  $Q$ s into  $P$ s rather than  $P$ s into  $Q$ s.)

c. The monopolist's profit,  $\Pi$ , is his total revenue,  $R$ , less his total cost  $C$ .

d. Suppose that his total costs are an increasing function of output  $C = c(Q, w)$  and with wage rates.

- e. Substituting for price and cost we obtain the monopolist's profit function:
- f.  $\Pi = R - C = Q d(Q, Y) - c(Q, w)$
- g. Differentiating with respect to  $Q$  and setting the result equal to zero generates a first order condition that characterizes the firm's profit maximizing output.
- h.  $\Pi_Q = R_Q - C_Q = d(Q, Y) + Q d_Q - c_Q = 0$  at  $Q^*$
- i. Or, doing a bit of subtraction,  $Q^*$  is such that  $d(Q, Y) + Q d_Q = c_Q$
- j. Again there is a nice economic interpretation of the first order condition. The first term,  $d(Q, Y) + Q d_Q$ , is marginal revenue, and the second,  $c_Q$ , is marginal cost.
- k. So the profit maximizing firm will produce at the output that sets marginal revenue equal to marginal cost.
- l. The implicit function theorem--about which we will talk more in the next lecture--allows  $Q^*$  to be written as a function of the other variables that are in the first order condition:  $Q^* = q(w, Y)$ .
- m. The price at which the output is sold can be found by substituting  $Q^*$  into the inverse demand function:  $P^* = d(Q^*, w)$ , and monopoly profit by substitution this and  $Q^*$  into the original profit function:  $\Pi^* = Q^* d(Q^*, Y) - c(Q^*, w)$ .
- iii. The substitution method can also be used, as in the first illustration, in cases in which the various relationships (functions) are assumed to take specific forms, as with the Cobb-Douglas, or CES, etc...
- iv. In general, the substitution method is a quite powerful technique for developing general models in settings in which a few key variables and constraints are focused on.

## II. Lagrangian Method

- A. The Lagrangian method uses a somewhat different method of modifying the objective function to account for (equality) constraints that restrict the feasible range of the control variables. (This method was widely used by theorists in the 1960s and 1970s, but has faded from the literature in the past two decades, although it remains in most Math Econ textbooks.)
- B. The Lagrangian method operates as follows:
- i. Transform all of the constraints into a form that equals zero.
    - For example, a budget constraint,  $W = P_1X_1 + P_2X_2$ , can be rewritten as:  $W - P_1X_1 - P_2X_2 = 0$ .
  - ii. Create a new objective function, called a *Lagrangian*, by multiplying each constraint (in its zero form) by a Lagrangian multiplier  $\lambda_i$  and adding the result(s) to the objective function.
    - For example, if the utility function is  $U = u(X_1, X_2)$  and the constraint is as above the Lagrangian would be:  $\mathcal{L} = u(X_1, X_2) + \lambda(W - P_1X_1 - P_2X_2)$
  - iii. Differentiate the Lagrangian with respect to all the control variables and the Lagrangian multipliers. (In the example, this would be with respect to variables  $X_1$ ,  $X_2$  and  $\lambda$ .)
    - The consumer does not control the other parameters of the optimization problem. The prices of the goods, and wealth are generally assumed to be exogenous at the level of the consumer. )
    - $\mathcal{L}_{X_1} = U_{X_1} - \lambda P_1$

- $\mathcal{L}_{X_2} = U_{X_2} - \lambda P_2$
  - $\mathcal{L}_\lambda = W - P_1 X_1 - P_2 X_2$
  - (**Subscripted variable names denote partial derivatives with respect to the variable subscripted.**  $\mathcal{L}_{X_1} = \delta \mathcal{L} / \delta X_1$ ,  $U_{X_1} = \delta U / \delta X_1$ , etc.)
- iv. At the constrained maximum (or minimum) all of these partial derivatives will equal zero.
- v. These first order conditions can be used to deduce properties of the constrained optimum in the same manner as the first order conditions of unconstrained optimization problems.
- a. For example, one can use the above to show that  $U_{X_1} / U_{X_2} = P_1 / P_2$ .
  - b. That is to say, at the utility maximum, the marginal rate of substitution between  $X_1$  and  $X_2$  equals their relative price.
- vi. In some cases, it is also necessary to determine whether the combination of control variables (the  $x$ 's) and Lagrangian Multipliers that satisfy the first order conditions also satisfies the second order conditions.
- This is only necessary if there is some doubt about the shape of the objective function and/or the feasible set.
  - To determine whether the second order conditions hold, requires differentiating the first order conditions with respect to the control variables and the Lagrangian multipliers, and then assembling the results into what is called a **bordered Hessian Matrix**.
  - (See the Matrix Magic handout.)
- vii. The Hessian Matrix techniques are rarely used in modern economics, because we usually assume that the objective function is concave and the feasible set is convex.
- (See the sufficiency theorems below in the Appendix, part IV, and in Chiang.)
- viii. The Lagrangian techniques are very useful when dealing with Cobb-Douglas functions and have been used in important papers, such as Samuleson's classic paper on public goods (R E Stat, 1954).  
[See the in-class notes for a derivation of an individual's demand curve from a CD utility function, using the Lagrangian technique.]

### III. Sufficiency Conditions (Look these up in Chiang.)

- A. Concave Programming: the Kuhn Tucker Sufficiency Theorem (Chiang 21.4, see also La Fuente Theorems 1.14, 1.18, 2.1, and 2.2).
- B. Quasi-Concave Programming: the Arrow Enthoven Sufficiency Theorem (Chiang 21.5, La Fuente Theorem 2.2).

### IV. Problem Set (Collected next week)

- A. Use the Lagrangian and Substitution Methods to characterize the utility maximizing level of goods  $G$  and  $H$  for three of the following cases:
  - i.  $U=HG$  where  $W = pG + H$ ,

- ii.  $U = H(G-2)^2$  where  $12 = G + 2H$ ,
- iii.  $U = HG$  where  $H^2 + G^2 = 20$
- iv.  $U = v(G) + u(H)$  where  $H + G = W$

**Next week:** the implicit function theorem and more applications and extensions of constrained optimization.

## V. Appendix: The Kuhn Tucker Approach: Non-Tangency Solutions and Inequality Constraints

A. Not all constrained optimization solutions are characterized by tangencies along an outer constraint of some kind, as with the classic consumer choice model.

- In many choice settings, there may be interior solutions or corner solutions.
- The latter, for example, occurs whenever a person purchases zero of some goods.
- These possibilities (corner and interior solutions) require different mathematical techniques, the most widely used of which are the Kuhn-Tucker techniques.
- It must be acknowledged that application of the KT technique have faded in the past twenty years and never have been as widely used as the Lagrangian techniques, but they are still used in cases in which tangency solutions cannot be taken for granted.

B. The Kuhn-Tucker first order conditions can characterize *corner solutions* and *"interior" solutions as well as the tangency solutions of the Lagrangian method*. *KT can also rule out negative solutions (that is to satisfy non-negativity constraints)*. It is, consequently, a somewhat **more general** optimization method than the other two methods discussed above. Unfortunately, it is also more difficult to apply.

- i. Corner solutions are solutions at the intersection (corner) of two constraints.
- ii. Interior solutions occur when a function has a local maximum within the feasible set (inside the boundary of the constraint set).
- iii. The Kuhn-Tucker method allows **non-negativity and inequality constraints** to be taken account of.
  - The setup for KT is surprisingly similar to the Lagrangian method in that it involves "adding" constraints to the objective function (the function being maximized or minimized) after writing them in a particular form.
  - [Use a consumer diagram to illustrate how non-negativity constraints work and cases in which consumers adopt corner or interior solutions to maximize their utility given their budget constraint.]

C. The Kuhn-Tucker Method generally entails the following steps:

- i. Convert all inequality constraints into an algebraic expression that is greater than or equal to zero.
- ii. Add each of these constraints to the objective function after multiplying each by its own unknown "Lagrangian" multiplier,  $\lambda_i$ , to form the Kuhn-Tucker objective function Z.
  - a. For example given utility function,  $U = u(X_1, X_2)$  and wealth constraint  $W \geq P_1X_1 + P_2X_2$  generate the Kuhn-Tucker function:
    - b.  $Z = u(X_1, X_2) + \lambda(W - P_1X_1 - P_2X_2)$
- iii. Differentiate function Z with respect to the control variables and the multipliers, as in the Lagrangian technique. (In the example, these are:  $X_1$ ,  $X_2$  and  $\lambda$ )
- iv. **At the constrained maximum**, the partial derivatives with respect the control variables may be less than or equal to zero, and those with respect to the "multipliers" may be greater than or equal to zero. The multipliers have to be greater than or equal to zero. The control variables are also usually assumed to be greater than or equal to zero. (See Chiang p. 726)
- v. If values for the control variables exist that set the partial derivatives of Z equal zero, the maximum occurs along the "upper" boundary, as with a Lagrangian.
- vi. At the corner solutions and within the feasible set, different conditions hold.
  - a. For example, the partial derivatives with respect to the control variables may be negative, indicating that the function is downward sloping as it approaches the upper bound of the feasible set.
  - b. In such cases, the constrained maximum occurs either inside the feasible set, or somewhere along its lower bound.
- vii. The **Kuhn Tucker conditions for the example are**:
  - a.  $U_{X_1} - \lambda P_1 \leq 0$  with  $X_1 \geq 0$  and  $X_1 (U_{X_1} - \lambda P_1) = 0$
  - b.  $U_{X_2} - \lambda P_2 \leq 0$  with  $X_2 \geq 0$  and  $X_2 (U_{X_2} - \lambda P_2) = 0$
  - c.  $W - P_1X_1 - P_2X_2 \geq 0$  with  $\lambda \geq 0$  and  $\lambda (W - P_1X_1 - P_2X_2) = 0$
- viii. The last term in the in the first order conditions, basically says that either your optimum lies along the constraints which bound from above (as in a Lagrangian) or along those which bound from below (along the non-negativity constraint), inside the feasible set, or both (as at a corner).
- ix. **Bliss points** and other maxima within the feasible set (constraints) can be characterized with the KT method. In that case,  $U_{X_1} = 0$  and  $\lambda = 0$ .

#### D. Optional KT Homework Problem

- i. Use the Kuhn-Tucker Technique to characterize the utility maximizing level of goods W and L when:
  - $U = L^{-5} + 1/(1+W)$  and  $24 \geq W + L$  with  $W, L \geq 0$
  - Interpret your result. What kind of "good" is W ?