

## Comparative Statics, Optimization, and the Implicit Function Theorems

Methodological individualism maintains that all "social" phenomena can be explained in terms of the behavior of individuals.

This approach to social science is one reason why economists and other social scientists use models of individual decision making as the basis of their models of social outcomes. For example, market demand curves are derived from individual demand curves which are themselves derived from constrained utility maximization by individuals. Supply curves are based on decisions reached by self-interested firm owners and input owners. Together supply and demand determine market price and output. Clearly, market prices and outputs will change if individual's make different decisions. Our models imply that such changes in plans may occur for a number of reasons.

In today's lecture, we develop tools for analyzing how individual (optimal) decisions change when parameters of their decision setting change, as when a consumer's wealth increases, the price of a substitute increases, etc. In most, but not all, of our models, the parameters of interest are ones in the constraints that bound the individual's opportunity set.

What micro economists refer to as "comparative statics" is essentially the study of how changes in parameters of individual choice problems affect individual choices, and thereby market (social) outcomes. The mathematics that describes how changes in parameters affect an individual consumer or firm's decision calculus is largely based on two closely related tools: (i) the implicit function theorem, and (ii) the implicit function differentiation rule.

### I. Elementary Comparative Statics

- A. Last week we found that an individual with a Cobb-Douglas utility function will have a demand function of the form:  $X_1^* = aW/P_1$ .
- i. Recall that we used the Lagrangian technique to derive this demand function.
  - ii. We formed a Lagrangian  $L = X_1^a X_2^{1-a} - \lambda (W - P_1 X_1 - P_2 X_2)$
  - iii. We differentiated with respect to the control variables  $X_1$ ,  $X_2$ , and the Lagrangian multiplier  $\lambda$ .
  - iv. We divided the first order conditions for  $X_1$  and  $X_2$  and did a bit of algebra to get  $X_2 = [(1-a)/a] [P_1/P_2] X_1$
  - v. We substituted this result into the budget constraint (for  $X_2$ ) and solved for  $X_1$  as a function of the other variables which gave us:  $X_1^* = aW/P_1$ .
- B. Given that result, we can see that as consumer wealth increases, such consumers will **always** increase their demand (for every price)
- i.  $d X_1^* / dW = a/P_1 > 0$  for all  $a > 0$  and  $P_1 > 0$ .
  - ii. In geometric terms, this consumer's demand curve shifts out to the right as his or her wealth increases.
  - iii. Note that this implies that all goods are normal goods, if consumers have Cobb-Douglas utility functions? ([Explain why.](#))

- C. We could also combine the demand curve with a supply curve to characterize market prices and quantities and use a bit of calculus to characterize how prices are affected by changes in supply and demand.
- i. To pick an easy example, suppose that  $X_1$  is land and that the supply of land is fixed at  $L^0$ . Assume also that there is just one consumer, who is a price taker.
    - At the price that sets supply equal to demand,  $X_1^* = L^0$
    - so,  $P_1^*$  must be such that  $aW/P_1^* = L^0$ .
    - A bit of algebra allows  $P_1^*$  to be solved for,  $P_1^* = aW/L^0$ .
  - ii. Note that we can easily generalize to the N-consumers case.
    - if we assume that all consumers are the same. In this case market demand is just  $L^D = NX_1^* = aNW/P_1$
    - In this case, the market clearing price will set  $L^0 = aNW/P_1^*$ ,
    - Which requires  $P_1^* = aNW/L^0$
- D. A bit of calculus allows us to see that prices for land always rise as consumer wealth increases, if N identical consumer have Cobb-Douglas utility functions
- i.  $dP_1^*/dW = aN/L^0 > 0$  for  $a>0$ ,  $N>0$ , and  $L^0>0$ .
  - ii. We can also see that as the number of consumers increases, prices necessarily rise in this case:
  - iii.  $dP_1^*/dN = aW/L^0 > 0$  for  $W>0$ ,  $a>0$ , and  $L^0>0$ .
  - iv. Note also that we can say that if land becomes more important to consumers (e.g., if  $a$  increases) that prices will also rise:
  - v.  $dP_1^*/da = NW/L^0 > 0$  for  $W>0$ ,  $N>0$ , and  $L^0>0$ .
  - vi. [Note also that these are all “other things being equal” results. Why?]
- E. This example illustrates how comparative statics can be done mathematically with a bit of calculus and algebra, whenever we know the functional form of utility, demand, supply, production functions, etc.
- F. The sufficiency theorems mentioned in the previous lecture demonstrate that:
- i. If the objective function is concave and the feasible set is convex, the first order conditions of our optimization problem completely characterize the optimum values of all control variables.
  - ii. However, it is often difficult to directly interpret "the solution" in an intuitive and meaningful way.
- G. Algebraic manipulation of the first order conditions can shed a bit of light on the properties of the solution characterized.
- i. For example, the first order conditions of a consumer choice model can be manipulated to show that the marginal rate of substitution between any two goods equals their relative prices.
  - ii. (This is the algebraic method of showing that the constrained utility maximum lies at a point where an indifference curve is tangent to the budget constraint.)
  - iii. Similar results can easily be found for production functions that demonstrate that profit maximizing firms will use inputs so that their marginal rates of technological substitution equal their relative prices.
  - iv. In many cases, these results allow comparative statics to be done on their associated Engles curves or output expansion paths.

## II. More Advanced Comparative Statics using the Implicit Function Theorems

- A. It bears noting, however, that there are many cases in which there will not be nice closed form solutions for the first order conditions or for the market phenomena of interest, even given specific functional forms for the relationships of interest.
- i. Moreover, also many cases in which we may not know the functional form of the relationships of interest, or want to make specific assumptions about functional forms.
    - In many cases, it will be EASIER to develop models with abstract functional representations of the relationships of interest.
    - **In such cases, comparative statics, surprisingly, can still be done!**
  - ii. But, in order to do so, we require two additional tools beyond ordinary calculus and algebra: namely the implicit function theorem and the implicit function differentiation rule.
- B. The implicit function theorem allows a variety of properties to be deduced from the first order conditions, including many that are useful for comparative statics.
- i. The **implicit function theorem** implies that the first order conditions to be used: to characterize the solution (optimal value of the control variable(s)) as a function of the parameters of the optimization problem.
  - ii. The **implicit function differentiation** rule allows the comparative statics of such functions to be derived.
    - a. That is to say, the "implicit function differentiation rule" can be used to describe how the "ideal values" of the control variables change as parameters of the choice problem change.
    - b. For example, in a consumer choice problem one can determine rate at which the utility maximizing quantity of a good changes as the consumer's income changes or as some price changes. ( $dQ^*/dY$ ,  $dQ^*/dP_s$ , ...) *The latter is the slope of the consumer's demand curve.*
- C. The **Implicit Function Theorem** (see Chiang 205 - 206, La Fuente 5.2): Given a function such that:
- $$F(Y, X_1, X_2, \dots, X_m) = 0$$
- i. Where the function  $F$  has continuous partial derivatives  $F_Y, F_{X_1}, F_{X_2}, \dots, F_{X_m}$ ,
  - ii. and at point  $(Y^o, X_1^o, X_2^o, X_3^o, \dots, X_m^o)$  satisfying condition i,  $F_Y$  is nonzero,
  - iii. *Then* there exists an  $m$ -dimensional neighborhood,  $N$ , of the point
 
$$(Y^o, X_1^o, X_2^o, X_3^o, \dots, X_m^o),$$
  - iv. in which an implicit function exists that characterizes each of the "ideal" values of the control variables of  $F$  as a function of non-control variables (the "parameters" of the choice problem):
 
$$Y = f(X_1, X_2, X_3, \dots, X_m)$$
  - v. This function satisfies  $Y^o = f(X_1^o, X_2^o, X_3^o, \dots, X_m^o)$  in particular, and more generally,  $Y = f(X_1, X_2, X_3, \dots, X_m)$  for all points within the neighborhood.
  - vi. This gives the function the status of an identity within neighborhood  $N$ .
  - vii. Moreover, the implicit function,  $f$ , is continuous and has continuous partial derivatives with respect to  $X_1, X_2, \dots, X_m$ .
    - a. In cases where there is a single first order condition, there is a fairly straightforward method by which these partial derivative can be computed.
    - b. (See the implicit function differentiation rule below.)
- D. The  $N$ -equation version of the implicit function theorem is broader in scope but essentially similar. (See Chiang 210 - 211, la Fuente 5.2.)

E. In microeconomic applications, the "zero function,"  $F(Y, X_1, X_2, \dots, X_m) = 0$ , is usually the first order condition of some optimization problem, and the implicit function is the individual's (or firm's) demand (or supply) function. However, the theorem applies to ANY function that equals zero.

F. The **implicit function differentiation rule** is used to characterize the partial derivatives of the implicit function "generated" as above.

i. The rule is surprisingly simple (la Fuente, theorem 2.1).

ii. The partial derivative of implicit function  $Y$  with respect to  $X_i$  is simply:

$$Y_{X_i} = F_{X_i} / -F_Y \quad (\text{where subscripts denote partial derivatives with respect to the variable subscripted}).$$

iii. The **derivation** of this rule relies on the total derivatives of  $F$ .

a. Recall that  $F(Y^o, X_1^o, X_2^o, X_3^o, \dots, X_m^o) = 0$

b. Thus the total derivative of  $F$  have to add up to zero.

$$dY F_Y + dX_1 F_{X_1} + \dots + dX_m F_{X_m} = 0$$

c. Consequently, if we allow only  $X_i$  and  $Y$  to vary,

$$dY F_Y + dX_i F_{X_i} = 0$$

d. Solving this expression for  $dY/dX_i$  yields:

$$dY/dX_i = F_{X_i} / -F_Y$$

iv. The implicit function differentiation rule allows one to characterize how the solution to an optimization problem varies as parameters of the problem vary.

G. Example: **Properties of an individual's "abstract" demand function**

i. Suppose that "Al" has a utility function,  $U = u(X_1, X_2)$  which is monotone increasing in  $X_1$  and  $X_2$ , twice differentiable and strictly concave. The latter may be assured by assuming that:

$$U_{X_1 X_2} \geq 0, \quad U_{X_1 X_1} < 0, \quad \text{and} \quad U_{X_2 X_2} < 0.$$

ii. Al wants to find the utility maximizing combination of  $X_1$  and  $X_2$  given the budget constraint that he faces,  $W = P_1 X_1 + P_2 X_2$ .

iii. Using the substitution method:  $U = u(X_1, (W - P_1 X_1)/P_2)$

iv. Differentiating with respect to  $X_1$  yields:  $U_{X_1} + U_{X_2} (-P_1/P_2) = 0$

a. The value of  $X_1$  that satisfies this first order condition will maximize utility.

b. Denote such that value of  $X_1$  as  $X_1^*$

v. Note that at  $X_1^*$ , the first order condition is a function like  $F$  in the definition of the implicit function theorem; that is to say, the "foc" always equals zero at  $X_1^*$ . Since the first order condition is differentiable (remember that we assumed that  $U$  was twice differentiable), an implicit function exists that characterizes  $X_1^*$  as a function of the other parameters of the choice problem.

$$X_1^* = x(W, P_1, P_2)$$

vi. Economists refer to this function as Al's **demand function** for  $X_1$ .

- vii. The effect of a change in the price of good 1,  $P_1$ , on Al's demand for good 1 can be characterized using the implicit function differentiation rule:
- $X_1^* \cdot P_1 = F_{P_1} / -F_{X_1}$
  - Given our first order condition in equation iv above,  $F_{P_1} / -F_{X_1}$  can be written as:

$$[ U_{X_1X_2} (-X_1/P_2) + U_{X_2} (-1/P_2) - U_{X_2X_2} (P_1/P_2)(-X_1/P_2) ]$$

$$-[U_{X_1X_1} + 2 U_{X_1X_2} (-P_1/P_2) + U_{X_2X_2} (-P_1/P_2)^2 ]$$

- viii. This expression is determined by carefully calculating the derivatives of the  $foc, U_{X_1} + U_{X_2} (-P_1/P_2)$ , with respect to  $P_1$  and  $X_1$ .
- H. The "sign" of this derivative of our implicit function tells us whether Al's demand curve slopes downward or not.
- The "sign" is jointly determined by all the partial derivatives in the expression above.
  - Most of these have already been characterized by our assumptions about Al's utility function and his budget constraint.
    - From the original characterization of  $U$  we know that all of the first partial derivatives are positive
    - We also know that all of the second derivatives are negative (This implies that both  $X_1$  and  $X_2$  are goods that exhibit diminishing marginal utility).
    - We also know that the cross partial is positive (an increase in good 2 increases the marginal utility of good 1).
    - Together these characteristics of  $U$  imply that Al's demand curve is **downward sloping**,  $X_1^* \cdot P_1 < 0$ .
    - (As an exercise, compute the derivative of the demand function with respect to Al's wealth and see whether "positive cross partials" also rule out inferior goods.)
- I. The implicit function approach has a wide range of applications in economic models and in game theory.
- (There are also multi-equation version of the theorem and differentiation rule, as developed in Chaining 8.5. These generalizations, however, are not very widely used.)

### III. Problem Set (Collected next week)

- A. Determine the first order conditions that characterize each of the following firm's profit maximizing level of output,  $Q^*$ , in the market setting characterized. Each firm produces its output at cost  $C$  and sells its output at price  $P$ :
- $C = w(Q^2 + 1)$  where  $P = P^0$  ("w" is the wage rate)
  - $C = c(Q, w, r)$  where  $P = P^0$  (Cost function  $c$  is monotone increasing and strictly *convex*, e.g. has positive second derivatives. "r" is the rental cost of capital.)
  - $C = c(Q, w, r)$  and  $P = p(Q, Y)$  (The cost function,  $c$ , is as before, and inverse demand function  $p$  is decreasing in  $Q$  and increasing in consumer income  $Y$ .)
- B. Use the implicit function theorems on first order conditions of part A to:
- write the ideal levels of output as a function of parameters of each firm's choice problem.
  - obtain derivatives of profit maximizing output levels with respect to wage rate  $w$ .

- iii. What, if anything, do your answers to part Bii imply about the short run effect of new labor laws that increase wage rates in the industry of interest?

**Next Lecture:** More on the implicit function theorem, and an introduction to time and uncertainty in economic decision making.

#### IV. Appendix: Comparative Statics and the Envelop Theorem

A. Another type of comparative static result involves how changes in some parameter of the choice problem (price of a substitute or input, wealth, change in technology, etc.) affects the best result (maximal utility, profit, etc.) of the decision maker.

- i. Many such properties seem intuitively obvious and often the math is fairly easy as well.
- ii. The **Envelop Theorem implies that derivatives of the “optimized objective function” are far simpler than one might expect.**
  - The optimized objective function (maximal utility, maximal profit, etc. functions) are simply the original objective function with the optimal values of the control values substituted in.
  - Normally these optimal values are themselves functions of parameters of the choice problem, as above with our demand functions.
  - Suppose that the objective function is:  $O = o(X_1, X_2, T)$  and there is a constraint that states that  $c(X_1, X_2, Z_1, Z_2, Z_3) = 0$ .
  - [“O” might be utility,  $X_1$  and  $X_2$  quantities of goods that the consumer can choose, and the constraint might be an abstract representation of a budget constraint, with the  $Z$ 's being relevant prices and consumer income.  $T$  would be a taster parameter of some kind.]
  - Suppose that you have solved the constrained optimization problem using either the substitution or Lagrangian technique and found that  $X_1^* = f(Z_1, Z_2, Z_3, T)$  and  $X_2^* = g(Z_1, Z_2, Z_3, T)$ , after taking partial derivatives, setting them equal to zero, and using the implicit function theorem.
  - The optimized objective function will be  $O^* = o(X_1^*, X_2^*) = o(f(Z_1, Z_2, Z_3, T), g(Z_1, Z_2, Z_3, T))$
  - The derivative of  $O^*$  with respect to  $T$  (a parameter of the original optimization problem, such as the price of a substitute or input) will be:
  - $O^*_T = O_{X_1} X_1^*_T + O_{X_2} X_2^*_T + O_T$
  - (This derivative,  $O^*_{Z_1}$ , tells you how the objective (utility, profits, etc.) is affected by changes in  $Z_1$  given that the decisionmaker is rational and is responding optimally to changes in  $Z_1$ .)
- iii. **This simplification is often useful mathematically and also often produces insights that are somewhat counter intuitive.**
  - The simplification occurs because many parts of the derivative will turn out to be terms from the first order conditions used to characterize the solutions of (or solve) the original optimization problem.
  - Many of the terms in the derivatives of  $O^*$  will all be zero along the “envelop” and so disappear from the derivative (e.g. they will necessarily have a value of zero).

- Normally--but not always--only the direct effects will remain. In such cases, the indirect effects--the terms including derivatives of  $X_1^*$  and  $X_2^*$ --turn out to equal zero or add up to zero along the “envelop,” and only terms like  $O_T$  will remain.
- In such cases,  $O^*_T = O_T$ , which implies that the effects of other “second order” adjustments are irrelevant for (small) changes in the parameter (T) of interest.

B. See Chaing and Wainwright 15.5 for a nice illustration of the “envelop effect” for a firm’s profit function.

### C. Illustration: the comparative statics for a classic consumer choice problem

- Suppose that Al choose  $X_1$  and  $X_2$  to maximize  $U = u(X_1, X_2)$  subject to  $W = P_1X_1 + P_2X_2$ .
- Solve for  $X_2$  as a function of  $X_1$ , etc. along the constraint,  $X_2 = [W - P_1X_1]/P_2$
- Substitute for  $X_2$  in the utility (objective) function:  $U = u(X_1, [W - P_1X_1]/P_2)$
- Differentiate with respect to  $X_1$  (the only control variable left) and set the result equal to zero to form a first order condition.
  - $U_{X_1} + U_{X_2}(-P_1/P_2) = 0$
- Appeal to the implicit function theorem to characterize the demand (optimized value) for  $X_1$ .
  - $X_1^* = f(P_1, P_2, W)$
  - Note that the implicit function differentiation rule can be used to characterize the properties (slopes) of Al’s demand function for  $X_1$ .
  - For example  $X_1^*_{P_1} = [U_{X_1X_2}(-X_1/P_2) + U_{X_2X_2}(-P_1/P_2)(-X_1/P_2) - U_{X_2}/P_2] / [-z]$
  - with  $[z] = U_{XX_{11}} + U_{X_1X_2}(-P_1/P_2) + U_{X_2X_1}(-P_1/P_2) + U_{X_2X_2}(-P_1/P_2)^2$
  - $[z]$  has to be negative if there is a unique solution (e.g. if the function with the substituted terms is strictly concave) [why?] **and if the second derivatives of U are negative and the cross partial are positive, this will be the case.** [Go through  $[z]$  term by term to see that this is true.]
  - $[z] < 0$  implies that  $[-z] > 0$ , so the sign of the derivative is characterized by the numerator.
  - Note that given our assumptions, every term in the numerator is less than zero, so Al’s demand curve is downward sloping! that is  $X_1^*_{P_1} < 0$ .
  - (This is not important for the envelop function illustration, but shows how one can use the tools of this lecture and the previous lecture to characterize and analyze consumer demand.)
- Substitute  $X_1^*$  into the solution above for  $X_2$  to characterize  $X_2^*$ .
  - $X_2^* = [W - P_1X_1^*]/P_2$
- Substitute  $X_1^*$  and  $X_2^*$  into either utility function above to characterize the maximum feasible utility for Al in this setting.
  - $U^* = u(X_1^*, [W - P_1X_1^*]/P_2)$
- The envelop theorem can be now be illustrated by take the derivative of this function with respect to one of the parameters of the choice problem ( $W$ ,  $P_1$ , or  $P_2$ ).
  - For example, the derivative with respect to  $W$  is:
  - $U^*_W = U_{X_1}X_1^*_W + U_{X_2}(-P_1/P_2)X_1^*_W + U_{X_2}/P_2$
  - (Note that this is not a first order condition, so we do not set it equal to zero.)

- We can reorganize this a bit by factoring the  $X_1^*w$  from the first two terms:
  - $U^*w = X_1^*w [U_{X_1} + U_{X_2}(-P_1/P_2)] + U_{X_2} / P_2$
  - Note that the term inside the brackets is always zero if we have maximized utility with respect to  $X_1$  and  $X_2$ .
  - [Why? Look at the derivative under “iv” above.]
  - Thus, the whole first term is zero along the “envelop” so,
  - $U^*w = U_{X_2} / P_2$  in this case.
- ix. The increase in utility generated by an increase in wealth is equal to the marginal utility of using the additional wealth to purchase ideal amounts of either good.**
- [Note that the derivative under “iv” above implies that  $U_{X_1}/P_1 = U_{X_2}/P_2$  .]