An Introduction to Non-Cooperative Game Theory

The use of game theory to analyze economic problems is the most active area of theory in modern economics. A quick look at any economics journal published (and many other social science and philosophy journals) in the past decade will reveal a large number of articles that use elementary game theory to analyze economic behavior in a variety of institutional and/or natural settings.

Moreover, it can be argued that the use of game theory in economics is older than game theory itself. For example, the Cournot duopoly model (1838) is an early example of a non-cooperative game with a Nash equilibrium. Stackelberg's (1934) analysis of duopoly is also game theoretic as are essentially all models of monopolistic competition. Game theory as a field in applied mathematics emerged after World War II.

Modern work on self-enforcing contracts, credible commitments, the private production of public goods, externalities, time inconsistency problems, models of negotiation, and models of political and social activity all use game theoretic models as their "engines of analysis."

Most of these applications apply the rational choice models (utility maximizing or net-benefit maximizing models) that we have used throughout this course. That is to say, the game theorists normally assume that players are "rational" in that the selection of strategies can be modeled as utility maximizing or net benefit maximizing choices. The result of the strategy choices of all the players in the game determine the payoffs (utilities or net benefits) actually received by the players.

In this introduction to game theory, we use rational choice models to analyze behavior in various game settings. The first settings analyzed involve relatively simple choice settings in which there are just two players and two possible strategies. Extensions of those foundational models involve shifts to settings with more possible strategies, larger numbers of players, and repeated games.

I. Non-Cooperative Games.

A. Game theory can be used to model a wide variety of human behavior in small number and large number economic, political, and social settings.

i. In "non-cooperative game theory" individuals are normally assumed to maximize their own utility without caring about the effects of their choices on other persons in the game.

- *The outcomes of the game, however, are usually jointly determined by the strategies chosen by all players in the game.*
- *Consequently, each person's welfare depends, in part, on the decisions of other individuals "in the game."*
- *Normally the payoffs are in terms of "utility," "euros," or "dollars," but occasionally other values are used.*

ii. For example:

- *In Cournot duopoly, each firm's profits depend upon its own output decision and also the output of the other firm in the market.*
- *After a snow fall, the amount of snow on a neighborhood sidewalks depends partly on one's own efforts at shoveling and partly that of all others in the neighborhood.*
- *In an election, each candidate's vote maximizing policy position depends in part on the positions of the other candidate(s).*

iii. Game theory models are less interesting in cases where there are no interdependencies.

- *For example, a case where there is no interdependence it that of a producer or consumer in perfectly competitive market.*
- *Here a consumer (or firm) is able to buy (or sell) as much as they wish without affecting market prices.*
- *Game theory can still be used in such cases, but with little if any advantage over conventional tools.*

B. The choice settings in which economists most frequently apply game theory are small number of player settings in which outcomes are jointly determined by players decisions that are made independently of one another.

- *That is to say, the most commonly used game theoretic models are small number non-cooperative games.*

C. The simplest game that allows one to model social interdependence is a two person game each of whom can independently choose between two strategies, S_1 and S_2.

i. There are four possible outcomes to the game:
- *(1) both players may choose S_1, (2) both may choose S_2*
- *(3) player A may choose S1 and player B may choose S2, or (4) vice versa.*

ii. The particular combination of strategies is the result of the independent decisions of the two players, A and B (Al and Bob).

D. For example, consider the "trading game" to the right.

i. Bob has bananas and Al has apples. Bob is thinking trading some bananas for some of Al's apples. Al is thinking about trading apples for some of Bob's bananas.

ii. Since trade is voluntary, nothing happens unless both players agree to trade. However, for the purpose of illustration, It is assumed that it costs "one util" a bit to make an offer, whether taken or not.

Bob

Al	**Trade**	**Don't**
Trade	**(a, b)** **(10,10)**	**(a, b)** **(1, 2)**
Don't	**(2,1)**	**(2,2)**

iii. Thus the lower left hand and upper right-hand cells have payoffs for Al and Bob that are lower for the one making the offer (trading), while the other is unaffected.

iv. (Games with such payoffs are sometimes called "**assurance game**s" or a Stag Hunt games, as discussed below.)

E. A game can be said to have a **Nash Equilibrium** whenever a strategy combination is “stable” in the sense that no player in the game has an interest in changing his or her strategy, given that of other players, because he or she cannot increase his or her payoff by doing so.

i. Note that the above trading game **has two Nash equilibria**, (trade, trade) and (don't, don't). Neither person can make themselves better off by changing their strategy (alone) given that of the other player(s) in the game in these cells (strategy combinations).

ii. A state of the world or game outcome is said to be **Pareto Optimal** or Pareto Efficient, if it is impossible to reach another state (outcome) where at least one person is better off and no one is worse off.

iii. Note that the (Trade, Trade), equilibrium is Pareto optimal, but not of the other outcomes are.

F. In economics, two person models of exchange, the Edgeworth Box, are often used to illustrate the principle of exchange between two persons, although we know that exchange in the real world is much more complex. Simple 2 person games have similar advantages.

i. 2 person - 2 strategy games can often capture the essential features of particular choice settings of interest to social scientists.

ii. This is also true of the Edgeworth box very nicely illustrates why the trade takes place and why a trade equilibrium tends to be Pareto optimal.

iii. However, as the above game theoretic representation of the "problem of exchange" demonstrates, simple 2x2 models nearly always neglect some details that may be important. (Again, this is a property of all models.)

II. The Prisoners' Dilemma Game: A Simple NonCooperative game.

A. The *Prisoners' Dilemma game* is probably the most widely used game in social science.

i. The "original" prisoners dilemma game goes something like the following. Two individuals are arrested under suspicion of a serious crime (murder or theft). Each is known to be guilty of a minor crime (say shoplifting), but it is not possible to convict either of the serious crime unless one or both of them confesses.

		Prisoner B	
		Testify	**Don't**
Prisoner A	**Testify**	**(10,10)**	**(1, 12)**
	Don't	**(12,1)**	**(2,2)**

ii. The prisoners are separated. Each is told that if he testifies about the other's guilt that he will receive a reduced sentence for the crime that he is known to be guilty of.

iii. The Nash equilibrium of this game is that BOTH TESTIFY (or Both CONFESS).

B. To see this, analyze the behavior that is likely to emerge given the payoffs represented in the matrix.

i. Each cell of the original Prisoner's dilemma game matrix contains payoffs for A and B, in **years in jail** (a bad).

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- *In generalized prisoner's dilemma (PD) games, the payoffs are nearly always in terms of utility, rather than years in jail or other tangible consequences, but tangible consequences can always be used, if one know how those consequences affect utility.*
- *When numbers are used as payoffs, they usually are intended to simply describe the rank order of the cells for the individual players.*
- *PD games are normally "symmetric games" in which players have the same strategy sets and payoff functions.*
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ii. Each individual prisoner will rationally attempt to minimize his jail sentence.

- *Note that regardless of what Prisoner B does, Prisoner A is better off testifying. 10 < 12 and 1 <2. Testifying is the dominant strategy.*
- *Note that the same strategy yields the lowest sentence for Prisoner B. If A testifies, then by also testifying B can reduce his sentence from 12 to 10 years. If A does not testify, than B can reduce his sentence from 2 to 1 year by testifying.*
- *Each player in a PD games has a* ***dominant strategy****, a strategy that yields the highest possible payoff, regardless of what the other player does.*

iii. The (testify, testify) strategy pair yields 10 years in jail for each.

- *This combination of strategies is a Nash equilibrium to this game, because each individual regards his own choice (testifying) to be optimal, given that the other player has testified.*
- *At a Nash equilibrium, no player can independently change his or her strategy in a manner that increases his or her payoff.*
- *Note that PD games have a single Nash equilibrium, but other games (such as the assurance game above) may have more than one Nash equilibria.*

iv. The Nash equilibrium of PD games is a "dilemma," because each prisoner would have been better off if neither had testified. (2 < 10).

- *The PD game demonstrates that independent rational choices do not always achieve Pareto optimal results.*
- *DEF: An outcome is Pareto Efficient if and only if there are not Pareto superior moves from that outcome.*
- *DEF: State X is Pareto superior to state Y, if and only if at least one person prefers X to Y and no one prefers Y to X.*
- *[f course, society at large may regard this particular "dilemma" if two dangerous criminals are punished for real crimes.*

C. The prisoner's dilemma game (PD game) can be used to illustrate a wide range of social dilemmas.

i. Competition between Bertrand (price setting) duopolists.

ii. Competition between Cournot (quantity adusting) duopolists.

iii. Decisions to engage in externality generating activities. (Pollution)

iv. Competition among students for high grades in universities

v. Commons Problems

vi. Public goods problems

vii. The Not In My Backyard (NIMBY) Problem

viii. The free rider problem of collective action.

ix. The international regulation dilemma

x. The arms race

xi. The Hobbesian Jungle

xii. A variety of team production problems

xiii. Contract Breach/Fraud (in settings with and without penalties)

xiv. (Several of these applications are illustrated below in these lecture notes.)

D. The essence of a PD game's dilemma is that the "cooperate, cooperate" solution is preferred by each player to the "defect, defect" equilibrium; but, nonetheless, the "defect, defect" outcome emerges from independent decision making.

i. This contrasts with independent choices in markets, where independent decision making tends to make the participants better off.

ii. For the "defect, defect" outcome to be a unique equilibrium, defecting has to yield a payoff that is a bit higher than the cooperative solution regardless of whether the other player cooperates or not.

iii. PD payoffs are often represented "ordinal" utility levels with (3, 3) for the mutual cooperative solution and (2, 2) for the mutual defection result. In such cases, the other payoffs are often represented as (1,4) and (4,1) with the defector receiving 4 and the cooperator 1.

iv. The PD payoffs can also be represented algebraically using (abstract) payoffs.

- *(C, C) and (D, D) are the payoffs of the mutual cooperation and mutual defection outcomes*
- *Let S be the "sucker's payof" and T be the "temptation payoff. (S, T) and (T, S) are thus the payoffs when one person defects and the other is "played for a sucker.*
- *In a PD game, $T>C>D>S$.*

E. The PD game's main limitations as a model of social dilemmas are its assumptions about the number of players (2), the number of strategies (2), the period of play (1 round).

- *However, these assumptions can be dropped without changing the basic conclusion of the analysis.*
- *Essentially the same conclusions follow for N-person games in which the players have a infinite number of strategies (along a continuum) and play for any finite number of rounds, as we will see later in the course.*

F. Another limitation of PD games is that they assume that players have to play them. They can not simply stop playing. The possibility of exit can also be analyzed with game matrices.

i. Such PDE games (prisoner dilemma games with exit) were examined by Vanberg and Congleton (1992) for cases in which not all players were "rational" in the sense that they always play the defect strategy when they participate in the game.

ii. They showed that in many such environments, conditional strategies of cooperation could be viable if the game is repeated many times.

iii. In such settings, relatively "nice guys" finished first rather than last. Using norms favoring cooperation in PDE settings can be viable strategies for individuals, rather than one that requires personal sacrifice.

G. Note that the **mathematical requirements for completely specifying a game** are met in the Prisoner's Dilemma game.

i. The possible strategies are completely enumerated

ii. The payoffs for each player are completely described for all possible combinations of strategies.

iii. The information set is (implicitly) characterized. (A player is said to have perfect information if he knows all details of the game. A perfectly informed player knows the payoffs for each party, the range of strategies possible, and whether the other players are fully informed or not.)

III. A Few Other "Named Games"

A. Several other interesting games are widely enough used to have been given names. Most can be created by changing the payoffs of the two player two strategies games..

i. A **zero sum** game is a game in which the **sum of the payoffs in each cell** is always zero. In this game, every advantage realized by a player comes at the expense of other players in the game.

- *(Individuals with no training in economics seem to regard all economic activities as zero sum games. Of course, in most cases, exchange creates value for each player. Trade is a positive sum game.)*

ii. **Coordination games** are games where the "diagonal" cells (top left or bottom right) have the same payoffs(for example, 1,1) which are greater than those in the off diagonal cells, (for example, 0,0).

- *Such games illustrate why it can be useful that a convention or norm is followed by both persons. Both players benefit if both do either strategy C or strategy D.*
- *For example, if we all drive on the left side of the road or all on the right, we all have higher payoffs than some drive on each side of the road.*
- *(Draw a sample game matrix for a coordination game and determine its Nash equilibria.)*

iii. **Assurance games** are similar to coordination games. The off diagonal payoffs for the "cooperative" strategy are equal to or below those of the on diagonal cells (2, 0), however the upper left-hand "cooperative" cell has a higher value to both players (3, 3) than the lower right-hand "do nothing" cell, the original position (2, 2).

- *It will take some kind of guarantee or trust to generate moves from the original lower right hand score to the higher upper left-hand cell.*
- *The trading game developed above is an assurance game.*
- *(Draw a sample game matrix for an assurance game and determine its Nash equilibria.)*
- *Some game theorists have renamed the assurance game a stag hunt game after a setting described by Rousseau in his Discourse on Income Inequality (1755)*

  "If it was a matter of hunting a deer, everyone well realized that he must remain faithfully at his post; but if a hare happened to pass within the reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple and, having caught his own prey, he would have cared very little about having caused his companions to lose theirs."

- *See Brian Skyrms' (2001) book on the Stag Hunt for a complete discussion.*
- *(Note that the above translation suggests that the Stag Hunt is really a PD game rather than an assurance game, at least if the starting point is two hunters at their stag hunting post. In the usual assurance game, the better equilibrium is, of course, stable!)*

iv. **Chicken** games are contests in which identical strategy choices are disastrous rather than beneficial.

- *In Chicken games, the "on-diagonal" strategies are lower than the "off-diagonal outcomes" yield higher payoffs.*
- *The classic chicken game involves two drivers driving in the dark down a country road, with each driver starting in the same lane.*

- *The person that changes lane is considered to be a "chicken".*
- *The off-diagonal payoff of the chicken is lower than that of the person who stays in his or her original lane, but normally higher than that received in the on-diagonal cells (e.g. the ones with both drivers do the same thing and crash).*
- *The off diagonal scores are generally higher, although one person does better than the other as with (4,2) and (2,4).*
- *(Create an illustrating payoff matrix for the dangerous 1950s teenage drivers game of chicken on rural roads.)*

IV. A Few Social Science Applications of PD-like Game Matrices

A. As noted above, a variety of social dilemma problems can be analyzing Prisoner's Dilemma Games.

i. One such game is the "Public Goods" or free rider problem.

ii. In this game, a public service can be produced by either player alone by paying the full cost of the service, or it can be jointly produced if each pay's for half of the service.

iii. (A **pure public good** is a good that is "perfectly shareable," a good which once produced can be enjoyed by all in the community of interest.)

iv. Suppose that the benefits from production of the service takes is simply $(C_A + C_B)P$ where C_A is Al's contribution, C_B is Bob's contribution and P is the productivity of the contributions. The net benefit is simply the benefit from the public service less the player's contribution.

- *If both players contribute C to the cost of the public good, then both players receive a* ***net benefits*** *(payoff) equal to 2CP - C.*
- *If neither player contributes, then the payoff for each is 0.*
- *If only one contributes, the person contributing receives PC-C and the person free riding receives PC.*
- *This structure of contributions and benefits yields a game with the following net benefit payoffs::*

v. Note that the magnitude of P (the productivity of contributions) is critical to whether there is a public good problem or not.

- *For there to be a problem, CP>2CP-C>0.*
- *A bit of algebra allows one to show that CP>2CP - C > 0 requires 1>P>0.5.*
- *(Prove this as a practice exercise.)*

A Public Good Game

		Bob	
		Contribute	Free Ride
Al	Contribute	(A, B) (2CP-C, 2CP-C)	(A, B) (CP - C, CP)
	Free Ride	CP, CP - C	(0,0)

vi. When P is in this range, the result is the classic Public Goods problem studied in Public Economics.

- *(Note that the payoffs in this case have the same rank order as those in a PD game, but in this case our simple production function motivates the payoffs. They are not simply assumed, but modeled.)*
- *The Nash Equilibrium of a game where 1<P<0 is mutual free riding, which generates the (0,0) outcome.*

vii. This example shows how a more realistic (detailed) choice setting can be represented using a game matrix.

- *Of course, in most cases, more than two persons will be involved in paying for or producing the public good, and in most cases the good itself can be produced at various levels.*
- *Still, the 2x2 representation, captures essential features of the "free rider" problem that must be confronted when thinking about the production of public goods.*

viii. It is interesting to note that if P> 1, no dilemma exists and the public good will be voluntarily provided.

B. The same matrix can be modified slightly to show **how rewards and penalties can be used to solve such problems**.

i. Assume that 1 < P < 0, in the above game.

ii. Suppose that free-riding can be observed, and that penalty F is imposed on any one that free rides.

- *The penalty could literally be a fine or a tax imposed on free riders.*
- *The penalty might also be nonpecuniary, as with losing the respect of approval of one's friends or neighbors.*
- *Or, the penalty could be entirely internal, as when a person that violates his or her private rules of conduct anticipates feeling guilty afterwards.*

iii. We now incorporate penalty F (a fine) into the game matrix.

- *Clearly F>C will solve the dilemma, but will other smaller fines also work?*

iv. How large does F have to be solve this public good problem?

- *To solve the problem, CP-F < 2CP - C and CP-C > -F (Explain why.)*
- *Show that the first requirement is satisfied when F> C-CP or F> (1-P)C*
- *Show that the second requirement is satisfied when the same condition is met.*
- *The smallest possible fine is thus F^{min} = (1-P)C + e where e is the smallest difference in payoffs that a player responds to (can observe).*

A Public Policy Solution to a Public Good Game

Bob

		Povide	Free Ride
Al	Provide	(A, B) (2CP-C, 2CP-C)	(A, B) (CP-C, CP-F)
	Free Ride	CP-F, CP - C	(-F, -F)

- *This implies that the smallest required fine falls with the productivity of contributions. [Why?]*
- *[Hint: Show this mathematically by differentiating F^{min} with respect to P]*

v. Similar matrices can be used to illustrate essential features of what economists call externality problems.

C. The above suggests that a wide variety of types of penalties can be used to solve PD dilemmas.

- *Some will involve formal legal sanctions: the use of police and courts to impose fines.*
- *Others may be solved by the creation of organizations that can impose modest punishments, as when managers may reduce the salaries of their employees, deny future raises, or fire free riders (shirkers).*

- *Some such penalties will be only stochastically imposed, which will require using expected values to model their effectiveness and calculating minimal fines (as in Tullock's Logic of the Law or Becker's classic paper on crime.)*
- *Culture and norms may also solve PD problems through by associating conditional "approval" with contributing to a public good (cooperation) and/or "shame" or loss of status with free riding (defection)*

V. Expanding the Strategy Set to Three or More options

A. Two Illustrations: the Regulatory Dilemmas of Neighboring Governments

B. Race to the Bottom. Suppose that are two communities that are interested in regulating some activity within their own territory.

C. Suppose further that regulations in each community affect each other's prosperity, with the community with the "weakest" regulations being somewhat more prosperous than the community with the stronger community.

i. To simply a bit, assume that there are just three types of regulations that can be imposed: weak, medium, and strong regulations.

ii. Suppose also that the joint ideal is "medium, medium"

iii. However, the effect of local regulations (relative to that of the other community implies that each community is a bit better off weakening its regulations, given the other's regulation of the activity of interest.

The Race to the Bottom Dilemma

Community B's Environmental Regulations

	weak	**medium**	**strong**
A's env regs	**A,B**	**A,B**	**A,B**
weak	**6,6**	**8,4**	**9,2**
medium	**4,8**	**7,7**	**8,5**
strong	**2,9**	**5,8**	**6,6**

iv. Such games have a Nash Equilibrium in Pure strategies that is not Pareto Efficient.

v. This "regulatory dilemma" is sometimes called the "Race to the Bottom" because each government has an incentive to under regulate the phenomena of interest (say air pollution).

vi. Notice also that a voluntary agreement to move to (medium, medium), as with a treaty, may not solve the dilemma because it is not a Nash equilibrium.

- *It is for this reason that treaties often, for example, have no effect on international air pollution [See, for example, papers by Murdoch and Sandler (1997)].*
- *This may be difficult to arrange in an international setting although it can be done within a federal system by higher levels of government.*
- *In a federal system, such problems can, however, be solved if higher levels of government punish (fine) communities that are free riding.*

D. NIMBY. Now suppose that the inter-community externality in the opposite direction. That is to say, suppose that the community with the weaker regulation attracts undesirable (say, noisy, ugly, or polluting industries) into the community.

i. Assume again that there are just three levels of regulation and that the two community ideal is (medium, medium) as in the previous example.

ii. In this case, each community is just a bit better off if it has somewhat tougher regulations than its neighbor.

iii. We can just slightly modify the payoffs of the above game to illustrate the new problem.

The Race to the Top Dilemma

NIMBY

Community B's environmental Regulations

	weak	medium	strong
A's env regs	**A,B**	**A,B**	**A,B**
weak	**6,6**	**4,8**	**2,9**
medium	**8,4**	**7,7**	**5,8**
strong	**9,2**	**8,5**	**6,6**

iv. This game also has a Nash Equilibrium with dominant strategies that is not Pareto Optimal.

v. This regulatory dilemma is sometimes called the "race to the top" or NIMBY (not in my backyard) problem.

VI. PD-like Games with Continuous Strategy Options

A. There are many settings in which players strategies are not discrete, but rather lie along a continuum of some sort.

- *Players on a team may work more or less.*
- *More or less of a public good may be provided.*

B. Such games can be represented mathematically by specifying a payoff (or utility) function that characterizes each player's payoffs as a function of the strategy choices of the players in the game of interest.

C. Consider, for example, a two-person lottery game in which each can buy as many tickets as they like and each player's probability of winning depends on the number of tickets owned by that player relative to the total sold. . Both want to maximize their "expected" net earnings from purchasing tickets.

i. The **expected value** of an event with outcomes 1, 2, i, ... N is $V^e = \Sigma$ PiVi, where Pi is the probability of event i, and Vi is the value of event i.

- *If Al purchases Na lottery tickets and Bob purchase Nb tickets, Al's expected profit is* $R_a^e = [Na / (Na + Nb)]Y - Na\ C$ *where Y is the prize one and C is the cost of a lottery ticket.*

- *Similarly Bob's expected net benefit (profit) is* $Rb^e = [Nb / (Na + Nb)]Y - Nb\, C$

ii. Al's expected profit maximizing number of lottery tickets can be found by differentiating Ra^e with respect to Na and setting the result equal to zero.

- $dRa^e/dNa = \{[1 / (Na + Nb)] - [Na / (Na + Nb)^2]\}Y - C = 0$ *at* Na^*
- *Putting terms over the same denominator and adding C to each side yields:*
- $[Na + Nb - Na]/(Na + Nb)^2 = C/Y$ *or* $Nb/(Na + Nb)^2 = C/Y$
- *Next we want to solve for Na*
- $Nb = (Na + Nb)^2 C/Y$ *or* $Nb(Y/C) = (Na + Nb)^2$
- *which implies that* $(NbY/C)^{1/2} = Na + Nb$
- *so* $Na^* = - Nb + (NbY/C)^{1/2}$

iii. This last function is sometimes called a **best reply function**. In this case, it tells Al the expected profit maximizing number of lottery tickets to purchase given any particular purchase by Bob.

- *Note that Na* varies with Bob's purchase which implies that Al does* ***not*** *have a dominant strategy.*
- *Note also that a best reply function can be derived for Bob,* $Nb^* = - Na + (NaY/C)^{1/2}$

iv. Note also that if both **persons are simultaneously on their best reply function**, neither can change their strategy and improve their payoff (remember that the best reply function for player i maximizes his or her payoff, given the strategies adopted by all other players), as required for the existence of a **Nash equilibrium**.

v. Thus, the **Nash equilibrium** of this lottery game occurs at a point where: $Na^* = - Nb^* + (Nb^*Y/C)^{1/2}$ and $Nb^* = - Na^* + (Na^*Y/C)^{1/2}$

- *To find the Na* and Nb* combination where both these conditions hold, one can either substitute the equation describing Nb* in terms of Na into the Al's best reply function and do a bit of algebra.*

vi. In a **symmetric game** (a game in which players have the same strategy sets and payoff functions) there is normally a symmetric equilibrium. In this case, the two best reply functions will intersect at a point where Na = Nb.

- *At the symmetric lottery game's equilibrium:* $Na = - Na + (NaY/C)^{1/2}$
  or $2Na = (NaY/C)^{1/2}$
- *Squaring both sides, we have:* $4Na^2 = NaY/C$ *which implies that* $4Na = Y/C$
- *or* $Na^{**} = Y/4C$ *and since Na = Nb at the symmetric Nash equilibrium, we also have* $Nb^{**} = Y/4C$

vii. Since each ticket costs C euros, so Al spends Na** C or Y/4 euros on tickets. That is he spend exactly 1/4 of the prize money (if he wins) on tickets.

- *[The same is true for Bob, so it is clear that this particular lottery will not be a "money maker" for its organizers.]*

D. The lottery game can be generalized to think about a wide variety of games in which one's odds of winning a contest depends upon how much time, energy, wealth, etc. one invests in the game.

E. Common applications include the political rent-seeking games, originally developed by Gordon Tullock (1980), legal battles in court, research and development contests by firms, warfare, car racing, grades on university exams, etc..

F. The lottery game and its various applications **can also be generalized** to take account of more than 2 players, and to include "technologies" where the exponents on investments are subject to increasing or decreasing returns.

G. It is surprisingly easy to generalize this game by, for example, including N players rather than two.

i. Let K represent the total investment of the N-1 players, then the expected payoff of a "typical" player is:
  - $Ra^e = [Na / (Na + K)]Y - Na\ C$

ii. Differentiating with respect to Na yields:
  - $dRa^e/dNa = \{[1 / (Na + K)] - [Na / (Na + K)^2]\}Y - C = 0$

iii. Solving for Na, as above, yields:
  - $Na^* = - K + (KY/C)^{1/2}$

iv. This equation is the **best reply function of a typical player** in the present N person game.

v. To find the symmetric equilibrium, note that K = (N-1) Na, so:
  - $Na^* = - (N\text{-}1) + [(N\text{-}1)Na\ Y/C]^{1/2}$

vi. solving for Na*, yields:
  - $Na^{**} = [(N\text{-}1)/ N^2]\ (Y/C)$

vii. Note that when N = 2, as above, Na** = (1/4) (Y/C) , as before.

viii. The **total expenditure** on "rent seeking" is NC times this amount, or (N-1)Y/N, and this expenditure approaches Y in the limit as N approaches infinity.

H. Different technologies for increasing one's chance of winning can also be taken into account by assuming changing our assumptions about investments in the game (Na) affect the probability of winning the prize. For example we can take account of economies and diseconomies of scale by changing from P =Na/(Na + K), to P = $Na^d/ (\Sigma\ Ni^d)$.

i. The payoff function for a typical player now becomes:
  - $Ra^e = [Na^d/ (\Sigma\ Ni^d)]Y - Na\ C$

ii. Differentiating with respect to Na now yields:
  - $dRa^e/dNa = \{[dNa^{d-1} / (\Sigma\ Ni^d)] - Na^d\ (dNa^{d-1}) / (\Sigma\ Ni^d)^2\}Y - C = 0$

iii. To find the symmetric equilibrium, note that Na = Ni for all i = 1, 2, ... N, so:
  - $\{[dNa^{d-1} / (NNa)] - Na^d\ (dNa^{d-1}) / (N^2 Na^2)\}Y - C = 0$, *or putting the numerators over a common denominator and collecting a few terms:*
  - $\{[dNNa^{2d-1} - dNa^{2d-1})] / (N^2 Na^{2d})\}Y - C = 0$, *or*
  - $\{[d\ (N\text{-}1)Na^{2d-1})] / (N^2 Na^{2d})\}Y - C = 0$

iv. solving for Na*, yields the individual's number of tickets (level of resources invested in the contest) at the symmetric Nash equilibrium:
  - $Na^{**} = [(N\text{-}1)/ N^2]\ (dY/C)$

v. Note that when d=1 and N=2, as above, Na** = (1/4) (Y/C) , as before.

- *However, the* ***total expenditure*** *on "rent seeking" is again NC times this amount, or d(N-1)Y/N.*
- *Note that total expenditures* ***will now exceed Y****, whenever d> (N-1)/N.*

I. To summarize:

i. The more players are in the game, the less each spends.

ii. However, the total spent rises with the number of players.

iii. In games with constant returns (the classic contest function) the total investment in the contest approaches the value of the prize (Y) as the number of players approaches infinity.

iv. Contests with increasing returns may have "super dissipation," where more resources will be invested in the contest than the prize is worth.

v. (Note that no player will routinely play such games. However, "no one" playing is also not an equilibrium, so potential players may play mixed participation strategies--more on that later in the course.)

J. There are a surprisingly large number of applications of these rent-seeking-lottery games.

- *Essentially any contest in which additional resources increases the probability of winning, or the fraction of the prize that is won, can be modeled with such functions.*
- *Indeed, a very large "contest" literature has emerged in the past ten or twenty years that explores such functions.*
- *To this point, the "Tullock" contest function has been most widely applied to represent interest group politics, although it can be used to represent crime, terrorism, etc. as noted above.*
- *Note that dissipation--the cost of the "competition"--is an important indicator of social welfare, particularly in contests that are "unproductive" and therefore wholly redistributive.*

K. Game theory can also be used to represent less concrete settings.

- *For example, payoff fuctions can be represented using abstract functions.*
- *And, equilibrium strategies can be characterized using a bit of calculus.*

L. Illustration: consider a *symmetric game in which each player has the same strategy set and the same payoff function.*

i. Suppose there are just two players in the game, Al and Bob.

- *Let the payoff of player A be G1 = g(X1, X2) and that of player B be G2 = g(X2, X1) where X1 is the strategy to be chosen by player 1 and X2 is the strategy chosen by player 2.*

ii. *Each player in a Nash game attempts to maximize his payoff, given the strategy chosen by the other.*

- *To find payoff maximizing strategy for player A, differentiate his payoff function with respect to* X_1 *and set the result equal to zero.*
- *The implicit function theorem implies that his or her best strategy* X_1^* *is a function of the strategies of the other player* X_2*, that is that* $X_1^* = x_1(X_2)$.
- *A similar reaction (or best reply) function can be found for the other player.*

iii. At the Nash equilibrium, both reaction curves intersect, so that $X_1^{**} = x_1(X_2^{**})$ and $X_2^{**} = x_2(X_1^{**})$

VII. Review Problems

A. Let R be the "reward from mutual cooperation," T be the "temptation of defecting from mutual cooperation," S be the "suckers payoff" if a cooperator is exploited by a defector, and P be the "Punishment from mutual defection." Show that in a two person game, relative payoffs of the ordinal ranking T > R > P > S are sufficient to generate a prisoner's dilemma with mutual defection as the Nash equilibrium.

B. Write down an assurance game and assume that the players initially find themselves at the less desirable Nash Equilibrium. Show that your trust problem can be solved by subsidies of various kind. Explain how this game differs from a PD game. Can subsidies also be used to solve a PD game?

C. Suppose that the inverse demand curve for a good is P = 100 - Q and that there are two producers. Acme has a total cost curve equal to C = 5Q and Apex has a total cost curve of C =10 Q. Each firm controls its own output. Prices are determined by their combined production. Characterize the Cournot-Nash equilibrium to this game.

D. Suppose that there are two neighbors, Ms 1 and Ms 2, each of whom enjoy playing their own music loudly enough to annoy the other. Each maximizes a utility function defined over other consumption, C, the volume of their own noise, and that of their neighbor's (a bad). Ms 1's utility function is $U_1 = C_1^{0.5} N_1^{0.5} N_2^{-0.5}$. Ms 2 has a similar utility function and each has a budget constraint of the form , Yi = Ci + Ni.

i. Characterize each neighbor's "best reply" or "reaction" function, and determine its slope.

ii. What happens to neighbor 1's reaction function if his income rises?

iii. Show the effect that a simultaneous increase in each neighbor's income has on the Nash equilibrium of this game.

iv. Is there anything strange about this game?