

I. Answers for the Problem Set from Lecture 2

A. Consider the demand function $Q = a + bP + cY$, with $b < 0$ and $c > 0$.

- i. Find the slope of this demand function in the $P \times Q$ plane.

Take the derivative $dQ/dP = b$

(Remember that when taking a partial derivative with respect to x , everything that is not a function of x , here a , c and Y , are treated as constants.)

- ii. Find the slope of this demand function in the $P \times Y$ plane.

Solve the equation for P as a function of Y : $P = (Q - a - cY)/b$

Take the derivative $dP/dY = -c/b$

- iii. Show that this demand function is homogeneous in prices iff $a = -cY$.

If and only if proofs have two parts. First we want to show if $a = -cY$, that the function $Q = a + bP + cY$ is homogeneous.

Note that in this case we can simplify the function to $Q = bP$

Let Q' be the value of the function at price P' . so $Q' = bP'$

Let Q'' be the value of the function at price KP' so $Q'' = bKP'$

Note that $Q''/Q' = bKP'/bP' = K$ which implies that $Q'' = KQ'$

Thus if $a = -cY$, the function becomes $Q = bP$, which is homogeneous of degree 1
 $Kf(P) = f(KP)$

Next we want to show that if a is not equal to $-cY$, that the function can not be homogeneous. [This is the hard part.]

To simplify notation, let $a - cY = Z$, so $Q = a + bP + cY = Z + bP$.

Let $Q' = bP' + Z$ and $Q'' = bKP' + Z$

If this function is homogeneous of degree g , then $Q''/Q' = K^g$,

$$\text{or } K^g = [bKP' + Z] / [bP' + Z]$$

which implies that : $K^g [bP' + Z] = [bKP' + Z]$

$$\text{and that } K^g [bP' + Z] - [bKP' + Z] = 0$$

gathering terms and factoring yields:

$$K(K^{g-1} - 1) bP' + (K^g - 1)Z = 0$$

Can this expression be true for an arbitrary $K > 0$, given bP' and Z different from zero? If the answer is no, then $Q = bP' + Z$ can not be homogeneous.

Note that $K(K^{g-1} - 1) bP' + (K^g - 1)Z = 0$

holds for an arbitrary bP' and Z different from zero only if both $(K^{g-1} - 1)$ and $(K^g - 1)$ equal zero, which is only true if $K = 1$. That is to say, the above equality cannot hold for an arbitrary K given an arbitrary Z and bP' different from zero!

QED

- iv. Is the associated revenue function ($R=PQ$) concave? strictly concave? (Hint: use the inverse demand function to characterize the price at which the firm can sell its output.)

The simplest way to determine concavity given the tools at our disposal is to compute the second derivative.

First find Price as a function of Q : $P = (Q - a - cY)/b$

This allows revenue to be written as:

$$R = PQ = [(Q - a - cY)/b] Q = (Q^2 - aQ - cYQ)/b$$

$$dR/dQ = (2Q - a - cY) / b$$

$$d^2R/dQ^2 = 2/b < 0 \text{ given } b < 0$$

(E. g. as long as a linear demand curve slopes downward the associated total revenue function will be strictly concave.)

- v. What is the revenue maximizing quantity of this good?

Revenue is maximized at the quantity where $R_Q = 0$, which we will call Q^*

For the present revenue function this occurs when $0 = (2Q^* - a - cY) / b$

Multiplying by b and isolating Q^* yields: $2Q^* = a + cY$

Thus revenue is maximized when $Q^* = (a + cY)/2$

(Surprisingly slope does not affect the revenue maximizing output! Moreover the revenue maximizing output for a linear demand curve is ALWAYS exactly half way to the point where the demand curve intersects the Q axis.)

- vi. Prove that this demand function is continuous in Y .

The proof is the same for each. Let the sequence Y_1, Y_2, \dots, Y_N have a limit point Y^* . Note that the sequence Q_1, Q_2, \dots, Q_N is $a + bP_0 + cY_1, a + bP_0 + cY_2, \dots, a + bP_0 + cY_N$.

Let $Q^* = (a + cY_0) + bY^*$

To determine whether Q^* is the limit point determine whether there are an infinite number of the Q series within any finite distance of $Q^* = (a + bP_0) + cY^*$.

Notice that $Q^* - Q_n = c(Y^* - Y_n)$ that is to say the distance between elements of the Q series is just " c " times the distance between the respective Y series.

If there are an infinite number of points within distance \mathbf{d} of Y^* then there are an infinite number of Q s within $c\mathbf{d}$ of Q^* . Since Y^* is a limit point of the Y series, \mathbf{d} can be made arbitrarily small and still encompass an infinite number of Y s, the same is true of an arbitrarily small $c\mathbf{d}$ for the Q series. QED

B. Use the substitution method to:

- i. Find the utility maximizing level of goods g and h in the case where

$$U = g^a h^b \text{ and } 25 = g + h,$$

Substituting for h yields $U = g^a (25-g)^b$

Differentiating with respect to g yields:

$$ag^{a-1} (25-g)^b + bg^a (25-g)^{b-1} (-1) = 0$$

Dividing by g^{a-1} and $(25-g)^{b-1}$ yields: $a(25-g) - bg = 0$

Isolating the g terms yields: $ag + gb = 25a$

Let the specific value of g that maximizes U , be referred to as g^*

Dividing yields $g^* = 25a/(a+b) = [a/(a+b)] 25$

(Note that $h^* = 25 - g^*$, so $h^* = 25 - [a/(a+b)] 25 = [b/(a+b)] 25$.)

This is a general solution for consumer choice problems where the consumers have a Cobb-Douglas utility function and can spend a fixed amount, here 10, between two goods, here g and h . It turns out that utility is maximized in such cases by allocating the "money" between the two goods in proportion to the exponents!

- ii. Find the utility maximizing bundle of goods when $50 = g + h$ (the wealth constraint is twice as high).

The method is the same as above. $g^* = [a/(a+b)] 50$

- iii. Characterize the profit maximizing output of a firm where $\Pi = pQ - C$ and $c = c(Q, w)$

No mention of specific demand curve is made, so we assume that the firm produces for a competitive market where p is the prevailing market price. In this case:

$$\mathbf{P} = pQ - c(Q, w)$$

Differentiating with respect to Q and setting the result equal to zero allows us to characterize the profit maximizing output of this firm as:

$p - C_Q = 0$ or $p = C_Q$. That is to say, the firm should find the output at which marginal cost C_Q equal price!

- iv. Is this Cobb Douglas utility function homothetic for $a > 0$ and $b > 0$? Prove.

Recall from the notes that all homogeneous functions are also homothetic, so if we can show this function is homogeneous we are done.

$$\text{Let } U_0 = g_0^a h_0^b$$

$$\text{and } U_1 = (z.g)^a (z.h)^b$$

$$\text{Dividing yields } U_1/U_0 = (z.g)^a (z.h)^b / g_0^a h_0^b$$

Factoring and combining the z terms in the numerator yields

$$U_1/U_0 = z^{a+b} g_0^a h_0^b / g_0^a h_0^b$$

$$\text{Therefore: } U_1/U_0 = z^{a+b}$$

The Cobb-Douglas family of utility functions are homogeneous of degree $a + b$, and are therefore also homothetic.