

## Answers for Problems Assigned Week 3

1. Use the Lagrangian and Substitution Methods to characterize the utility maximizing level of goods G and H in the case where

### A. Substitution Method

- i. maximize  $U = HG$  where  $W = pG + H$
- ♦ substituting yields:  $U = (W - pG)G = GW - pG^2$
  - ♦ differentiating with respect to G yields:  $W - 2pG = 0$  (at  $G^*$ )
  - ♦ solving for  $G^*$  yields  $G^* = W/2p$
- ii.  $U = H(G-2)^2$  where  $10 = G + 2H$
- a. Note that  $G = 10 - 2H$
- ♦ substituting yields:  $U = H(8-2H)^2$
  - ♦ differentiating with respect to H yields:
  - ♦  $U_H = (8-2H)^2 + 2H(8-2H)(-2) = 0$  (use the composite function and multiplicative rules.)
  - ♦  $U_H = 4H^2 - 32H + 64 - 32H + 8H^2 = 0$  (at  $H^*$ )
  - ♦  $U_H = 12H^2 - 64H + 64 = 3H^2 - 16H - 16 = 0$
- b. The quadratic formula implies
- ♦  $H = [16 \pm \sqrt{(256 - 192)}] / 6$
  - ♦ so:
  - ♦  $H^* = 16/6 \pm \sqrt{8}/6 = \{ 4/3, 4 \}$
  - ♦ and
  - ♦  $G^* = 10 - 2H^*$  or  $\{ 22/3, 2 \}$
- iii.  $U = HG$  where  $H^2 + G^2 = 10$
- a. Note that constrain implies  $H = \pm \sqrt{10 - G^2}$
- ♦ so  $U = \pm G \sqrt{10 - G^2}$
- b. Differentiating with respect to G yields
- ♦  $(10 - G^2)^{1/2} + (1/2)G(10 - G^2)^{-1/2}(-2G) = 0$  at  $G^*$
- c. Multiplying by  $(10 - G^2)^{1/2}$  to simplify yields:
- ♦  $(10 - G^2) - G^2 = 0$  or  $2G^2 = 10$
  - ♦ so  $G^* = \pm \sqrt{5}$  (plus or minus the square root of 5)

d. Substituting  $G^*$  into the equation characterizing H as a function of G yields;

- ♦  $H^* = \pm \sqrt{10 - G^{*2}} = \pm \sqrt{5}$

### B. Lagrangian Method

- i.  $U = HG$  where  $W = pG + H$
- ♦ Form a Lagrangian:  $L = HG - \lambda(W - pG - H)$
  - ♦ differentiating yields:  $H - \lambda p = 0$
  - ♦  $G - \lambda = 0$
  - ♦ and:  $W - pG - H = 0$
  - ♦ Now solve for H, G, and  $\lambda$ .
  - ♦ dividing the first two yields:  $H/G = p$  or  $H = Gp$
  - ♦ which implies that  $W - pG - pG = 0$  or  $G^* = W/2p$
  - ♦ and  $H = W/2$
- ii. Given:  $U = H(G-2)^2$  where  $10 = G + 2H$
- ♦ Form a Lagrangian:  $L = H(G-2)^2 - \lambda(10 - G - 2H)$
  - ♦ Differentiate with respect to H and G (the control variables) and  $\lambda$  which yields three equations:
  - ♦  $L_H = (G-2)^2 - 2\lambda = 0$
  - ♦  $L_G = 2GH - 4H - \lambda = 0$
  - ♦  $L_\lambda = (10 - G + 2H) = 0$
- a. Next, solve this system of simultaneous equations for H and G.
- ♦ Dividing a/b and solving for G yields  $G = 4H + 2$
  - ♦ Substituting into the constraint for G yields
- $$10 = 4H^* - 2 + 2H^* \text{ or } H^* = 8/6 = 4/3$$
- ♦ and since  $G^* = 10 - 2H^*$  (from the constraint)  $G^* = 22/3$
  - ♦ (You have to be on the constraint under Lagrange. If you had just substituted back into  $G = 4H + 2$  you would be describing a point on the consumption expansion path. Both methods should yield the same result, because the solution is where the consumption expansion path crosses the budget line.)
- iii. Given  $U = HG$  where  $H^2 + G^2 = 10$

- ♦ Form a Lagrangian:  $L = HG + \lambda(10 - H^2 - G^2)$
- ♦ Differentiate with respect to control variables H and G and the Lagrangian multiplier  $\lambda$
- ♦  $L_H = G - 2H\lambda = 0$
- ♦  $L_G = H - 2G\lambda = 0$
- ♦  $L_\lambda = 10 - H^2 - G^2 = 0$
- a. Solve this system of simultaneous equation (c, d, e) for G and H. (You can also solve for the Lagrangian multiplier if you wish, but this is optional.)
- ♦ Dividing c and d, then cross multiplying yields  $H^2 = G^2$
- ♦ or  $G = +/- H$
- ♦ Substituting into the constraint for G yields:  $H^2 + H^2 = 10$
- ♦ or  $H^* = +/- (5)^{1/2}$
- b. That is to say,  $H^*$  equals plus or minus the square root of 5.
- c. Since from "c,"  $G^* = +/- H^*$   $G^*$  also equals  $+/- (5)^{1/2}$

2. Use the Kuhn-Tucker Technique to characterize the utility maximizing level of goods W and L when:

- i.  $U = L^{.5} + 1/(1+W)$  and  $24 \geq W + L$  with  $W, L \geq 0$
- ♦ Note that U is strictly concave and the feasible set is convex. The second derivatives of U are both negative, and the cross partial is greater than or equal to zero.
- a. Form the KT function in a manner similar to that of the Lagrangian:
- ♦  $K = L^{.5} + (1+W)^{-1} + \lambda(24 - W - L)$
- b. Differentiating with respect to the control variables W and L (work and leisure) and with respect to  $\lambda$  yields the following KT first order conditions:
- c.  $K_L = .5 L^{-.5} - \lambda \leq 0$  with  $L \geq 0$  and  $L(.5 L^{-.5} - \lambda) = 0$
- d.  $K_W = -(1+W)^{-2} - \lambda \leq 0$  with  $W \geq 0$  and  $W(-(1+W)^{-2} - \lambda) = 0$
- e.  $K_\lambda = 24 - W - L \geq 0$  with  $\lambda \geq 0$  and  $\lambda(24 - W - L) = 0$

- ♦ Before working on the KT conditions, take a moment to think about the objective function. Note that utility rises with L and *falls with W*. W is a bad. This person will consume as little as possible of the bad, W, and as much as possible of the good, L. The constraints determine what is possible.
- f. Now consider whether  $W = 0$  and  $L = 24$  satisfies the KT conditions.
- ♦ Equation e, the constraint, is clearly satisfied--as an equality.
- ♦ Equation d, the f. o. c. for W is also clearly satisfied. The negative of a squared term is always negative, unless it equals zero. With  $W = 0$ , we are still OK as long as  $\lambda$  is positive or a small negative number we are all right here. In that case,  $W=0$  satisfies the non-negativity constraint and the product constraint.
- ♦ Equation c is satisfied only if  $\lambda$  takes the value  $.5(24)^{-.5}$  in which case the  $K_L = 0$ . If  $L = 24$ , then L is clearly greater than zero, and our assumption about  $\lambda$  assures that  $K_L = .5 L^{-.5} - \lambda = 0$ . Given this value for  $\lambda$ , the product constraint is satisfied.
- ♦ We have now shown that the combination  $W = 0, L = 24, \lambda = .5 L^{-.5}$  satisfy the KT first order conditions.
- ♦ QED
- ♦ Given U quasi concave, and the convex opportunity set, the combination  $W = 0, L = 24, \lambda = .5(24)^{-.5}$  satisfies the KT first order conditions for a maximum
- ♦ (The Arrow Enthoven sufficiency theorem implies that given a convex feasible and quasi-concave objective function, the KT first order conditions are sufficient to characterize the constrained optimal values of the control variables.)
- ii. Interpret your result. What kind of "good" is W?
- ♦ As noted above W is a bad, perhaps unpleasant and unpaid work in a setting where leisure is an option. Consequently, this individual chooses a corner solution with  $L = 24$  and  $W = 0$ .