Answers for Problems Assigned Week 7 + +

I. Review Problems from Lecture 7

A. Suppose that Al is a "grade maximizing" student who is attempting to allocate her time between studying for two courses at the end of the semester. Her expected grade in either course increases as hours devoted to studying that course increase, but at a diminishing rate. She can only study for one course at a time. She has S hours to allocate.

i. Develop an objective function and constraints for analyzing Al's time allocation problem.

a. $G = g(T_1, T_2)$ where $S = T_1 + T_2$

b. with $G_{T1} > 0$, $G_{T2} > 0$, $G_{T1T1} < 0$, $G_{T2T2} < 0$, and $G_{T1T2} > 0$

c. [Keep the model as simple and general as seems sufficient to address the problem of interest. Here the student's grade point average rises as more time is devoted to studying in each course, T_1 and $T_2 > 0$, but at a diminishing rate, $G_{T1T1} < 0$ and $G_{T2T2} < 0$. The latter together with $G_{T1T2} > 0$ will guarantee that the objective function is strictly concave.]

ii. Find the first order condition(s) that characterize her grade maximizing allocation of time across the courses.

a. Using the substitution method, $G = g(T_1, S - T_1)$, differentiating yields:

b. $G_{T1} - G_{T2} = 0$ at T_1^*

iii. Use the implicit function theorem to generate a "studying supply" curve for one of the courses modeled. How does studying time for this course change as S changes?

a. Given ii. b., $T_1^* = t(S)$

b. To find how time spent studying for course 1 changes as S changes, apply the implicit function differentiation rule:

c. $T_1{}^*{}_S = H_S / -H_{T1} = (G_{T1T2} - G_{T2T2}) \,/\, -(G_{T1T1} - 2G_{T2T1} + G_{T2T2}) > 0$

d. [Note that given the assumed shape of the grade point average production function, that the numerator is unambiguously positive, as is the denominator as a whole. Consequently, in this model, time devoted to the first class unambiguously increases as total time available for studying increases in this case.]

B. Suppose that N risk neutral competitors participate in a lottery game with a fixed prize. Each player may purchase as many tickets as he wishes. The prize is awarded to the player whose name is drawn from a barrel containing all of the tickets. So, the expected prize for player i is $R[\, t_i / (t_i + t_o)\,]$, where R is the value of the prize, t_i is the number of tickets purchased by player i, and t_o is the number purchased by all other players.

i. If tickets cost C dollars each, find the number of tickets that maximizes player 1's expected reward for a given purchase by all other players.

a. The expected net benefits of purchasing t_i lottery tickets is

- $N^e = R[\, t_i (t_i + t_o)^{-1}\,] - t_i C$

b. Differentiating with respect to t_i yields:

- $N^e{}_{ti} = R[\, 1 / (t_i + t_o) - t_i / (t_i + t_o)^2\,] - C = 0$ at t_i^*

c. Note that "b" can be solved for t_1:

- $R\, ((t_i + t_o) - t_i) = C\, (t_i + t_o)^2$ (multiplying both sides by $(t_i + t_o)^2$)
- $R\, t_o / C = (t_i + t_o)^2$ (gathering terms and dividing by C)
- $t_i^* = - t_0 + (R\, t_o / C)^{1/2}$ (taking the square root of both sides and solving for t_i^*)

ii. Find a similar "best reply" function for some other player.

- $t_j^* = - t_0 + (R\, t_o / C)^{1/2}$ (consider Mr j.)

iii. An equilibrium may be said to arise in this game when no player has a reason to alter his ticket purchases. Find (or characterize) the equilibrium to this game when N=2, when N=3, when N=n.

a. The case for two players can be solved for directly in a manner similar to the general case which is done below.

b. The other cases are easier to solve when one generalizes a bit.

c. Note that the "best reply" functions are all very similar. (As it turns out, this is a *symmetric game*.) Consider the case where there are N-1 other players, and all players behave similarly in equilibrium. In this case, $t_i^* = t_j^*$ for all i, j = 1, 2...N.

- In this case, t_o can be written as $(N-1)\ t_i^*$

d. Substituting for t_o in our best reply function from "i. c." yields:

- $t_i^* = -(N-1)\ t_i^* + (R\ (N-1)\ t_i^* / C)^{1/2}$

e. This can be solved for t_i^* (The particular solution we will denote t_i^{**}.)

- $Nt_i^* = ((N-1)\ t_i^*\ R / C)^{1/2}$ (Adding (N-1) t_i^* to each side and simplifying)
- $N^2 t_i^{*2} = (N-1)\ t_i^*\ R / C$ (Squaring both sides)
- $t_i^{**} = ((N-1)/N^2)\ (R/C)$ (Dividing by t_i^* and N^2)

f. Note that in the two person case this becomes:

- $t_i^{**} = ((2-1)/2^2)\ (R/C) = (1/4)\ (R/C)$

g. and in the three person case:

- $t_i^{**} = ((3-1)/3^2)\ (R/C) = (2/9)\ (R/C)$

h. [There will be more problems like this in the second half of the semester as we move into analysis of game theory.]

BONUS---solution to the Kuhn - Tucker problem on the Study Guide

C. Suppose that Mr. Workaholic's utility function is $U = Y^2$ and that he faces the following constraints: Y = (24-L)20, L ≥ 6 and Y ≥ 10. Find Mr. W's utility maximizing level of leisure, L.

a. Substitute for Y and set up the K-T optimand:

b. $Z = [(24-L)20]^2 + \lambda_1 (6 - L) + \lambda_2 (10 - (24-L)20)$

c. Differentiate Z with respect to L, λ_1, and λ_2 which yields the following first order conditions:

- $Z_L = 40(L - 24) - \lambda_1 + \lambda_2 (20L) \le 0$ with $L \ge 0$ and $L\ Z_L = 0$
- $Z_{\lambda 1} = 6 - L \ge 0$ with $\lambda_1 \ge 0$ and $\lambda_1 Z_{\lambda 1} = 0$
- $Z_{\lambda 2} = 10 - (24-L)20 \ge 0$ with $\lambda_2 \ge 0$ and $\lambda_2 Z_{\lambda 2} = 0$

d. Note that the geometry of the problem suggests that the solution will be L = 6, $\lambda_1 \ne 0$ and $\lambda_2 = 0$.

- This follows because Mr. Workaholic does not value leisure at all and so will maximize income given his leisure and income constraints.
- He can take no less than 6 hours of leisure, at which point Y = (24-6) 20 = 360 > 10. So this solution is feasible.
- (Also, remember that the constraints that bind have "multipliers" (λ's) that differ from zero, while those that do not bind have "multipliers" (λ's) equal to zero.)

e. Note that $\lambda_2 = 0$ will satisfy the third first order solution and L = 6 will satisfy the second first order solution.

f. Note also that at L = 6, [40 (6 - 24)] = (-720) Consequently, if $\lambda_1 = (-720)$ the first of the first order conditions will be satisfied as an equality ($Z_L = 0$).

- Therefore, the solution L = 6, $\lambda_1 = -720$, and $\lambda_2 = 0$ satisfies the Kuhn Tucker first order conditions.
- Since the objective function is strictly concave and the constraints are convex, either of the sufficiency theorems mentioned in Chaing implies that the KT first order conditions fully characterize the solution(s) to the original optimization problem.
- QED

(A suggestion ß)

Some General Advice About Approaching Problems:

For those who are a bit nervous about the exam/study guide problems, here is some general advice about how to approach the problem part of the exam (or of model building in general):

i. First, determine what kind of problem is being posed: (i) optimization, (ii) intertemporal, (iii) uncertainty, (iv) comparative statics, or (v) changes in an optimized variable.

ii. Second, set up the problem. This is the most important step.

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- Most economic models concern choices.
- Who is making the choice?
- What is his, her or their goal in economic terms? What is the objective function (utility, profit, ...)?
- What are the constraints? Are there continuous control variables (such as quantity(s), time, probability) or is it a binary choice (buy or not, sell or not, hire or not, enter or not, all or nothing)?
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iii. Third, use the right tools:

- (i) calculus (substitution method) for optimization,
- (ii) present value for intertemporal choice,
- (iii) expected value for problems involving uncertainty,
- (iv) implicit function differentiation rule for comparative statics, or
- (v) envelop theorem for changes in an optimized function (profit or utility).
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- [Most problems of interest in this course will involve a sequence of i followed by iv, and occassionally v. Also some optimization problems and problems involving uncertainty may involve more than one approach: uncertainty and optimization, or present value and uncertainty.]

iv. Fourth, turn the crank.

- By now you should all be able to get the first order conditions right once the model is properly written down.
- Mostly, the first order conditions involve using the composite function and multiplication rules in the models that we use the substitution method to solve.
- For the comparative statics (implicit function derivatives) just be sure to remember that every place there is a variable/term in the original objective function (after the substitution) there is now a similar term in the partial derivatives. REMEMBER: the partial derivatives are themselves functions.
- That means that the derivatives of EACH of the partial derivatives functions in the first order conditions resemble those of the original objective function. If there are two terms that would emerge from differentiating the objective function, there will be two terms for the derivatives of EACH of the partial derivatives of that function.
- Just be patient, and don't panic if there are a lot of terms.
- Determine the signs of each term by referring to the model's initial assumptions--which normally are based on economic theory and intuitions: e. g. goods have positive marginal utility but diminishing marginal utility, inputs have positive marginal product but diminishing marginal product, cross partials are usually positive, etc. (These assumptions are generally sufficient to imply strict concavity for the objective function.)
- Some times you can catch math errors by appraising your answer with your initial economic intuition. If a sign seems wrong, there may well be a math error. However, always be prepared to learn something new from your model.

v. Fifth, recheck the above.

vi. Evaluate your results.

- Does the answer make sense?
- Is there anything new here?
- Can your model be improved?