I. An Introduction to Mathematical Methodology.

Scope of Course, Usefulness and limitations of deductive methodology
Basic Concepts: compact sets, convexity, continuity, functions,
Application: preference orderings, fundamentals of consumer theory.

A. Essentially all scientific work attempts to determine what is general about the world.

i. For example, successive sun rises may be more or less beautiful but all sun rises on Earth are caused by the Earth's daily rotation in combination with light generated by our nearest star..

ii. Persons take account of many different characteristics of an automobile when they select a car, but all at some point all must consider the price of the car. Economics argues that the higher is the price the less likely a given person is to purchase a particular car, other things being equal.

B. This process of finding relationships or general rules of thumb is complex but may itself be described as a joint exercise in logic (model building) and observation (empirical testing).

i. This course attempts to provide students with the core mathematical concepts and tools that are the most widely used by economists in the "model building" part of the scientific enterprise.

ii. Econometrics will introduce you to the core statistical methods used in the "testing" part of economic science. For the most part, these tools are mostly involve the concepts and methods of mathematical optimization, and various methods for applying them to settings where rational decisions are, or can be, made.

C. Not all models are mathematical, but mathematical models have may advantages over other modeling methods. Two of the most important are LOGICAL CONSISTENCY (what you have derived is true, given your assumptions) and CLARITY (you know, or should know, what you have assumed).

i. When the rules of logic are applied to numbers the result is mathematics. Most of the mathematics we have been taught can be deduced from a few fundamental assumptions using the laws of logic. (See the postulates of Peano, an Italian mathematician (1850 - 1932).)

ii. Economists often attempt to deduce some consequences of human decision making from fewest fundamental assumptions that may be expected to characterize "optimizing man" and the circumstances under which his choices are made.

iii. This general logical and mathematical enterprise does help explain some of the core intuitions and esthetics of economists. This deductive and **reductionist methodology** is often central to the efforts of economic theorists to understand human behavior.

II. Some Fundamental Concepts and Definitions from Mathematics

A. Some fundamental properties of **relationships**:

i. DEF: Relationship R is **reflexive** in set X, if and only if aRa whenever a is an element of X.

ii. DEF: Relationship R is **symmetric** in set X if and only if aRb then bRa whenever a and b are elements of set X.

iii. DEF: Relationship R is **transitive** in set X if and only if aRb and bRc then aRc when a, b, and c are elements of set X.

iv. Recall that within the set of real numbers, there are relationships which are symmetric (equality), reflexive (equality) and transitive (equality, greater than, less than, greater than or equal than, less than or equal than).

v. In economics there are also several relationships which possess all three properties, and some that exhibit only transitivity.

vi. In general, economists assume that preference orderings satisfy all three of these properties. Indeed, *rationality in microeconomics is often defined as transitive preferences.*

B. DEF: A **function** from set X to (or into) set Y is a rule which assigns to each x in X a unique element, f(x), in Y. Set X is called the domain of function f and set Y its range.

C. DEF: A **utility function** is a function from set X into the real numbers such that if aPb then U(a) > U(b) and if aIc then U(a)=U(c) for elements of the set X.

i. Note that the indifference relationship, I, can be defined in terms of the weak preference relationship R.

a. The weak preference relationship R means "at least as good as."

b. Note that if aRc and cRa, then aIc.

ii. Similarly, the strong preference relationship, P, can be defined in terms of the weak preference relationship.

a. The strong preference relationship means "better than."

b. Note that if aRb but b~Ra then aPb.

D. Note that the assumption that a utility function exists, is equivalent to the assumption that individual preferences are such that: preferences are transitive;

i. preferences are also complete: each bundle (combination of goods and/or "bads") has a unique rank.

a. Every bundle of goods generates either more or less or the same utility level as other goods.

b. (Some theorists make a distinction between complete and incomplete utility mappings from X to R, but this distinction is not important for "routine" decisions.. Why?)

E. Some important definitions and concepts from Set Theory.

i. DEF: An infinite series, x1, x2, ... xn is said to have a **limit** at x* whenever for any d >0, the interval x* - d, x* + d contains an infinite number of points from the series. (That is to say, x* is a limit point of a series in any case where there are an infinite number of elements of the series arbitrarily close to x*.)

ii. DEF: A set is **closed** if it contains all of its limit points.

iii. Def: A set is **bounded** if every point in A is less than some finite distance, D, from other elements of A.

iv. Def: A set is **compact** if it is closed and bounded.

v. Def: A set is **convex** if for any elements X_1 and X_2 contained in the set, the point described as $(1-\alpha)X_1 + \alpha X_2$ is also a member of the set, where $0<\alpha<1$.

a. Essentially a convex set includes all the points directly between points in the set.

b. That is to say, any convex (linear) combination of two points from the set will also be a point in the set.

c. Thus a solid circle, sphere, or square shaped set is a convex set but not a V-shaped or U-shaped set. What other common geometric forms are convex?

d. Example from economics: usually "better sets" are assumed to be convex sets. That is to say, the set of all bundles which are deemed better than bundle a is generally assumed to be a convex set.

e. Another example, is the budget set, the set of all affordable commodities give a fixed wealth and fixed prices for all goods that might be purchased.

F. Convexity and compactness assumptions are widely used in models of human decisionmaking. For example, opportunity sets and production possibility sets are nearly always assumed to be convex and compact.

G. Some important concepts and definitions from **Calculus**

i. Def: Function Y = f(X) is said to be **continuous** whenever the limit of f(X) approaches Y= f(Z) as X approaches Z.

a. Or alternatively, function Y = f(X) is said to be continuous if for every point in the domain of X, and for any e >0, there exists d > 0, such that $|f(X) - f(Z)| < e$ for all X satisfying $|Z - X| < d$.

b. (That is to say, points only a finite distance from Z should generate function values within a finite distance of f(Z). In fact, f is continuous if for any finite distance e (epsilon) there exists d (delta) such that any value within delta of z generates a function value within epsilon of f(z).)

ii. Def: the limit of a function: function f is said to have a **limit point** y* at x* if and only if (iff) for every e > 0, there is a d > 0 such that $|f(x) - y^*| < e$ whenever $|x - x^*| < d$.

a. The limit of f at x* is denoted $\lim_{x \Rightarrow x^*} f(x) = y^*$

b. If there is a real number y* satisfying this definition at x*, we say that the limit of f at x* exists.

c. Note that this definition rules out the existence of different right hand and left hand limits. (why?)

iii. Def: function f is said to be **differentiable** if and only if (iff) for every x contained in set X the limit point of $\{ [(f(x) - f(z)]/(x - z) \}$ exists.

iv. Note that if f is differentiable, f is also continuous. (why?)

H. Within microeconomics, utility functions and production functions are generally assumed to be continuous and twice differentiable.

i. Such assumptions clearly rule out some kinds of *decision makers* just as the assumption that production possibility sets and opportunity sets are convex and compact rule out some kinds of *choice settings*.

ii. These assumptions are made largely for "economic" rather than "empirical" reasons. That is to say, generally it is felt that the benefits of more tractable models overwhelms the costs of reduced realism and narrower applicability.

iii. Of course if continuous versions of the choice settings lead to empirically false predictions, then continuity assumption should be dropped.

a. When discrete aspects of the choice problem are, or may be, important, various tools from set theory, integer programming, and real analysis can still be applied.

b. Its just that in most of the cases of interest to economists, the assumption of continuity is approximately correct. There may be a smallest grain of sand, but it is pretty small!

c. Try to think of cases where the simplifying assumptions of continuity and convexity will generate predictions about behavior that are clearly wrong.

III. Problem Set (Collected next week)

A. Suppose that Al always prefers larger apartments to smaller ones, but is unable to discern difference of ten sq. ft. or less. Are Al's preferences transitive? Explain.

B. Determine whether the following sets are convex sets or not.

i. Al has a budget set W > PaA + PbB, where A and B are both non negative numbers. Pa is the price of good A and Pb is the price of good B. Is Al's budget set convex?

ii. Barbara has a bliss point "B" characterized in terms of all goods relevant over which her preferences are defined. Consider bundle "C" which has less of every good than B. Is the better set for "C" convex? (Construct two two-dimensional examples.)

C. Consider the function $f(X) = 1/X^2$ for $X \neq 0$ and $f(X) = 1$ for $X = 0$.

i. Is the domain of f compact?

ii. Is f a continuous function?

iii. Is f monotone increasing for x>0 ?

iv. Prove that the f does not have a limit at X=0.

D. For most purposes, economists assume that utility functions are continuous and twice differentiable.

i. What does this differentiability imply about the commodity space over which the utility function is defined?

ii. What does twice differentiability imply about the shape of the utility function?

iii. Are their any important limitations of models which rely upon the assumption of differentiable utility functions? Discuss.

iv. Is differentiability an important modeling assumption or a mathematical convenience? Explain.

Next week: optimization, concavity, and rational choice