

## An Introduction to Optimization

The core of modern economics is the notion that individuals optimize. That is to say, individuals use the resources available to them to advance their own personal objectives as well as they know how. In consumer theory, the consumer's decision is represented as an effort to maximize utility given a budget set. In the theory of the firm, firms are assumed to maximize profit, given production technology and market opportunities. In public choice theory, candidates are assumed to maximize votes given the positions of other candidates and the preferences of voters.

Such choices can all be characterized with the mathematics of constrained optimization in settings where the individual controls something that may be more or less continuously varied: time, money, policy position, output. In each case there is an objective (utility, profit, votes ...) and each case there are constraints which characterize a feasible set (budget set/line, production function, distribution of voter preferences ...).

Today's lecture develops necessary and sufficient conditions for the existence of a maximum and minimum (with and without constraints) and the use of derivatives in demonstrating the concavity of a function.

### I. More Fundamental Concepts and Definitions from Mathematics

A. Many of the mathematical properties of a given function can be deduced from its "shape." One of the most widely used notions of shape is *concavity*. Here are three notions of concavity:

i. DEF: Strictly Concave: function  $f$  is *strictly concave* iff

$$\alpha f(X_1) + (1-\alpha)f(X_2) < f(\alpha X_1 + (1-\alpha)X_2) \quad \text{where } 0 < \alpha < 1.$$

(That is, iff points on a cord connecting two points on a function lie beneath the function.)

ii. DEF: Concavity: function  $f$  is *concave* iff

$$\alpha f(X_1) + (1-\alpha)f(X_2) \leq f(\alpha X_1 + (1-\alpha)X_2) \quad \text{where } 0 \leq \alpha \leq 1.$$

(Points on a cord connecting two points of a function lie beneath or on the function.)

iii. DEF: Quasi-Concave: Concavity: function  $f$  is *quasi concave*

$$\text{iff } f(X_1) \leq f(\alpha X_1 + (1-\alpha)X_2) \quad \text{where: } 0 \leq \alpha \leq 1$$

and  $f(X_1) < f(X_2)$ .

(The function lies above, or not below, the lower of the two end points of a cord connecting points on the function.)

iv. Any strictly concave function is also a concave function, but not vice versa.

v. Any concave function is also a quasi concave function, but not vice versa.

B. Economic analysis generally models both individuals and firms as maximizers. Individuals maximize utility. Firms maximize economic profit. As it turns out, in most cases there are theoretical and/or empirical reasons to believe that both utility functions and profit functions are concave. Concave functions have a number of useful properties in the context of "maximizing" behavior.

i. A *strictly concave* function has at most one local maximum.

ii. A **concave function** may have an infinite number of global maxima, (a horizontal line is concave), but if there is more than one maximum, they make up a continuous interval.

iii. DEF: The **local maximum** of a function,  $f(x)$ , has a value which exceeds those of other points within a finite neighborhood of  $x^*$ . That is,  $f(x^*)$  is a local maximum if  $f(x^*+e) < f(x^*)$  and  $f(x^*-e) < f(x^*)$  for  $0 < e < E$ , for some  $E > 0$ .

iv. DEF: The **global maximum** of a function,  $f(x)$ , has a value which exceeds all others over the entire range of the function (e.g. for every neighborhood of  $x^*$ ).

C. Derivatives of functions can be used to characterize *sufficient* conditions for local maxima and minima, as well as concavity.

i. A differentiable function is at a **local maximum or minimum** whenever its first derivative has the value zero.

ii. Given that one is at an extremal (an  $f(x)$  where the first derivative is zero),

a. the extremal is a local maximum if the second derivative is negative,

b. the extremal is a local minimum if the second derivative is positive.

iii. A function is *strictly concave* if its second derivative is negative over its entire domain.

iv. A function is *concave* if its second derivative is less than or equal to zero over its entire domain.

v. Note that a strictly concave function has at most a single maximum.

vi. A matrix of partial derivatives, called a *Hessian*, can be used to determine whether a multidimensional function is concave. At a local maximum, the Hessian will be negative definite.

D. Some Useful Derivative Formulas

i.  $y = ax + C$   $dy/dx = a$

ii.  $y = ax^b + C$   $dy/dx = bax^{b-1}$

iii.  $q = ax^by^c$   $dq/dx = abx^{b-1}y^c$

iv.  $u = \log(x)$   $du/dq = 1/x$

v.  $h = y(x)z(x)$  product rule  $dh/dx = (dy/dx)(z) + (y)(dz/dx)$

vi.  $h = y(z(x))$  composite function rule  $dh/dx = (dy/dz(x))(dz/dx)$

E. To increase the generality of their models, economists generally use very abstract functions characterized only by partial derivatives or concavity.

- i. Generally economists *assume* that many of the functions of interest (utility functions, profit functions, production functions) are twice differentiable and concave over the range of interest.
  - ii. (Assuming strict concavity is more or less equivalent to assuming that the functions of interest exhibit diminishing marginal returns.)
- F.** Occasionally, for the purposes of illustration, or in order to mathematically derive a specific functional form for an economic relationship of interest ( a demand curve, cost function, etc.), further structure is assumed.
- i. For example, a utility function might be assumed to have an exponential or *Cobb-Douglas* form,  $U = ax^b y^c$ , where:  $x$  and  $y$  are quantities of two goods. In a Cobb-Douglas function the exponents sum to one,  $b + c = 1$ .
  - ii. At other times somewhat less restrictive assumptions are used. For example a production function might be assumed to be **homogeneous** of degree 1.
    - a. DEF. A function is said to be *homogeneous of degree k*, if and only if whenever  $Y = f(X)$ , then  $f(\lambda X) = \lambda^k Y$
    - b. A production function that is homogenous of degree 1 exhibits constant returns to scale. Doubling all inputs doubles outputs.
    - c. Cobb-Douglas functions and linear functions through the origin ( $Y = ax$ ) are homogeneous of degree 1.
  - iii. Occasionally, utility and production functions are assumed to be *homothetic*, a somewhat more general family of functions than homogeneous functions.
    - a. DEF. A **homothetic function** is a composite function of the form  $H = h(Q(a,b))$  where  $Q$  is a homogeneous function and  $dH/dQ \neq 0$ .
    - b. All homogenous functions are homothetic functions but not all homothetic functions are homogeneous.
    - c. Homothetic utility functions have linear income expansion paths. Similarly, homothetic production functions have linear output expansion paths. (*The slopes of the isoquants are the same along any straight line through the origin.*)

## II. Constrained Optimization

- A.** Many optimization problems facing individuals and firms are constrained in some manner. In such cases, the above "unconstrained" maximizing (or minimizing) technique *can not* be directly applied -- by either the individuals themselves or those attempting to model their behavior. (For example, utility maximizing individuals are constrained by a budget constraint.)
- B. The Substitution Method.** In some cases, it is possible to "substitute" the constraint into the objective function (the function being maximized) to create a new composite function which fully reflects the effect of the constraint.
- i. For example consider the separable utility function:  $U = x^5 + y^5$  to be maximized subject to the budget constraint  $100 = 10x + 5y$ . (Good  $x$  costs 10 \$/unit and good  $y$  costs 5 \$/unit. The consumer has 100 dollars to spend.)

- ii. Notice that, from the constraint we can write  $y$  as:
 
$$y = [100 - 10x]/5 = 20 - 2x$$
- iii. Substituting into the objective function (in this case, a utility function) yields a new function entirely in terms of  $x$ :  $U = x^5 + (20 - 2x)^5$
- iv. Note that this function accounts for the fact that every time one purchases a unit of  $x$  one has to reduce his consumption of  $y$ .
- v. Differentiating with respect to  $x$  allows the utility maximizing quantity of  $x$  to be characterized:
 
$$d[x^5 + (20 - 2x)^5]/dx = .5 x^{-.5} + .5(20-2x)^{-5}(-2)$$
  - a. This derivative will have the value zero at the constrained utility maximum.
  - b. Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for  $x$  yields:
 
$$4x = 20 - 2x \Rightarrow 6x = 20 \Rightarrow x^* = 3.33$$
  - c. Substituting  $x^*$  back into the budget constraint yields a value for  $y^*$ 

$$y = 20 - 2(3.33) \Rightarrow y^* = 13.33$$
- vi. No other point on the budget constraint can generate higher utility than that at  $(x^*, y^*) = (3.33, 13.33)$ .

## III. Problem Set (Collected next week)

- A.** Consider the demand function  $Q = a + bP + cY$ , with  $b < 0$  and  $c > 0$ .
- i. Find the slope of this demand function in the  $QxP$  plane.
  - ii. Find the slope of this demand function in the  $PxY$  plane.
  - iii. Show that this demand function is homogeneous in prices iff:  $a = -cY$ .
  - iv. Is the associated revenue function ( $R=PQ$ ) concave? strictly concave? (Hint: use the inverse demand function to characterize the price at which the firm can sell its output.)
  - v. What is the revenue maximizing quantity of this good?
  - vi. Prove that this demand function is continuous in  $Y$ .
- B.** Use the substitution method to:
- i. find the utility maximizing level of goods  $g$  and  $h$  in the case where  $U = g^a h^b$  and  $20 = g + h$ ,
  - ii. find the utility maximizing bundle of goods when  $40 = g + h$  (i.e. if the wealth constraint is twice as high).
  - iii. characterize the profit maximizing output of a firm where  $\Pi = pQ - C$  and  $c=c(Q, w)$ .

**Next week:** more on constrained optimization: Lagrangian and Kuhn-Tucker methods, sufficiency theorems.