Essentials of Constrained Optimization

Whenever a person uses scarce resources to advance his or her goals, the decision can be represented as an exercise in constrained optimization. A person at the grocery store tries to find the shopping basket full of food that best satisfies his or her appetite and health. A busy person allocates time between paid and unpaid work, travel, study, and recreation in order to realize the best life that he or she can imagine, given the time and opportunities at hand. A firm allocates its R&D budget among alternative projects to maximize expected profit.

Because so many business, engineering, and political decisions are constrained optimization problems, a good deal of mathematical research has been applied to developing alternative mathematical methods for solving these problems. As a result, there is both a vocabulary of optimization and an array of optimizing techniques.

The language of optimization distinguishes between **objective function**s (the goals: profit, utility, etc.), **constraint**s (which describes what is feasible), control variables (the variables that can be directly determined by the decision maker) and parameters (variables that can not be controlled by the decision maker, or at least are taken as given for the present decision). [DEF: the **feasible set** is the collection points that satisfies the constraints of the problem of interest.]

There are **three mathematical optimization methods** that are widely used by economists: the substitution method, the Lagrangian method, and the Kuhn Tucker Method. Today's lecture will review these methods and discuss the conditions that are sufficient to fully characterize the optimal allocation of scarce resources among alternative uses.

I. The Substitution Method

A. In some cases, it is possible to "substitute" the constraint(s) into the objective function to create a new "composite function" that fully represents the effects of the constraint on the original objective function.

i. Generally, the substitution method attempts to reduce the number of control variables that have to be taken account of.

ii. The substitution method generally entails the following steps:

a. First., use the constraints to completely specify of the control variables as functions of the subset of control variables that are of most interest.

b. Second, substitute these relationships into the objective function to form a new objective function (that reflects all the constraints).

c. Third, differentiate with respect to each of the control variables that remain. (Often this will be just a single variable.)

d. If the new objective function is strictly concave, the optimal value(s) of the control variable(s) given the constraint(s) is the one(s) that sets the first derivative equal to zero.

iii. For example consider the separable utility function: $U = x^{.5} + y^{.5}$ to be maximized subject to the budget constraint $100 = 10x + 5y$. (Good x costs 10 $/unit and good y costs 5 $/unit. The consumer has 100 dollars to spend.)

iv. We can rewrite the constraint as $y = [100 - 10x]/5 = 20 - 2x$

v. Substituting this for Y in the objective function (here, the utility function) yields a new function *entirely in terms of x*: $U = x^{.5} + (20 - 2x)^{.5}$

vi. This new function accounts for the fact that every time one purchases a unit of x one has to reduce his consumption of y. (Why?)

vii. Note also that this new objective function has just one control variable, x.

viii. Differentiating with respect to x and setting the result equal to zero allows the utility maximizing quantity of x to be characterized:

$$d[x^{.5} + (20 - 2x)^{.5}]/dx = .5x^{-.5} + .5(20-2x)^{-.5}(-2) = 0 \quad \text{(at U max)}$$

a. The derivative will have the value zero at the constrained utility maximum. Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for x yields:

$$4x = 20 - 2x \Rightarrow 6x = 20 \Rightarrow x^* = 3.33$$

b. Substituting x* back into the budget constraint yields a value for y*

$$y = 20 - 2(3.33) \Rightarrow y^* = 13.33$$

ix. No other point on the budget constraint can generate a higher utility level than that $(x^*,y^*) = (3.33,13.33)$.

II. Lagrangian Method

A. The Lagrangian method uses a somewhat different method of modifying the objective function to account for (equality) constraints that restrict the feasible range of the control variables. The method operates as follows:

i. Transform all of the constraints into a form that equals zero. For example, a budget constraint, $W = P_1X_1 + P_2X_2$, can be rewritten as: $W - P_1X_1 - P_2X_2 = 0$.

ii. Create a new objective function, called a *Lagrangian*, by multiplying each constraint (in its zero form) by a Lagrangian multiplier λ_i and adding the result(s) to the objective function. For example, if the utility function is $U = u(X_1, X_2)$ and the constraint is as above the Lagrangian would be:

$$L = u(X_1, X_2) + \lambda(W - P_1X_1 - P_2X_2)$$

iii. Differentiate the Lagrangian with respect to all the control variables and the Lagrangian multipliers. (In the example, this would be with respect to variables X_1, X_2 and λ . The consumer does not control the other parameters of the optimization problem. The prices of the goods, and wealth are generally assumed to be exogenous at the level of the consumer.)

a. $L_{X1} = U_{X1} - \lambda P_1$

b. $L_{X2} = U_{X2} - \lambda P_2$

c. $L_\lambda = W - P_1X_1 - P_2X_2$

d. (**Subscripted variable names denote partial derivatives with respect to the variable subscripted**. $L_{X1} = \delta L/\delta X_1$, $U_{X1} = \delta U/\delta X_1$, etc.)

iv. At the constrained maximum (or minimum) all of these partial derivatives will equal zero.

v. These first order conditions can be used to deduce properties of the constrained optimum in the same manner as the first order conditions of unconstrained optimization problems.

a. For example, one can use the above to show that $U_{X1} / U_{X2} = P_1 / P_2$.
b. That is to say, at the utility maximum, the marginal rate of substitution between X_1 and X_2 equals their relative price.

vi. In some cases, it is also necessary to determine whether the combination of control variables (the x's) and Lagrangian Multipliers that satisfy the first order conditions also satisfies the second order conditions. This is only necessary if there is some doubt about the shape of the objective function and/or the feasible set. To determine whether the second order conditions hold, requires differentiating the first order conditions with respect to the control variables and the Lagrangian multipliers, and then assembling the results into what is called a bordered Hessian Matrix. (See the Matrix Magic handout.)

vii. This technique is rarely used in modern economics, because we usually assume that the objective function is concave and the feasible set is convex. (See the sufficiency theorems below in part IV.)

III. Kuhn Tucker

A. The Kuhn-Tucker method allows with **non-negativity and inequality constraints** to be taken account of. It is surprisingly similar to the Lagrangian method in that it involves "adding" constraints to the objective function (the function being maximized or minimized) after writing them in a particular form.

B. Kuhn-Tucker can characterize *corner solutions and "interior" solutions as well as the tangency solutions of the* Lagrangian method. *KT can also rule out negative solutions.* It is, consequently, a somewhat **more general** optimization method than the other two methods discussed above. Unfortunately, it is also more difficult to apply.

i. Corner solutions are solutions at the intersection (corner) of two constraints.

ii. Interior solutions occur when a function has a local maximum within the feasible set (inside the boundary of the constraint set).

C. The Kuhn-Tucker Method generally entails the following steps:

i. Convert all inequality constraints into an algebraic expression that is greater than or equal to zero.

ii. Add each of these constraints to the objective function after multiplying each by its own unknown "Lagrangian" multiplier, λ_i, to form the Kuhn-Tucker objective function Z.

a. For example given utility function, $U = u(X_1, X_2)$ and wealth constraint $W \geq P_1X_1 + P_2X_2$ generate the Kuhn-Tucker function:

b. $Z = u(X_1, X_2) + \lambda(W - P_1X_1 - P_2X_2)$

iii. Differentiate function Z with respect to the control variables and the multipliers, as in the Lagrangian technique. (In the example, these are: X_1, X_2 and λ)

iv. **At the constrained maximum**, the partial derivatives with respect the control variables may be less than or equal to zero, and those with respect to the "multipliers" may be greater than or equal to zero. The multipliers have to be greater than or equal to zero. The control variables are also usually assumed to be greater than or equal to zero. (See Chiang p. 726)

v. If values for the control variables exist that set the partial derivatives of Z equal zero, the maximum occurs along the "upper" boundary, as with a Lagrangian.

vi. At the corner solutions and within the feasible set, different conditions hold.

a. For example, the partial derivatives with respect to the control variables may be negative, indicating that the function is downward sloping as it approaches the upper bound of the feasible set.
b. In such cases, the constrained maximum occurs either inside the feasible set, or somewhere along its lower bound.

vii. The **Kuhn Tucker conditions for the example are**:

a. $U_{X1} - \lambda P_1 \leq 0$ with $X_1 \geq 0$ and $X_1 (U_{X1} - \lambda P_1) = 0$
b. $U_{X2} - \lambda P_2 \leq 0$ with $X_2 \geq 0$ and $X_2 (U_{X2} - \lambda P_2) = 0$
c. $W - P_1X_1 - P_2X_2 \geq 0$ with $\lambda \geq 0$ and $\lambda (W - P_1X_1 - P_2X_2) = 0$

viii. The last term in the in the first order conditions, basically says that either your optimum lies along the constraints which bound from above (as in a Lagrangian) or along those which bound from below (along the non-negativity constraint), inside the feasible set, or both (as at a corner).

ix. **Bliss points** and other maxima within the feasible set (constraints) can be characterized with the KT method. In that case, $U_{X1} = 0$ and $\lambda = 0$.

IV. Sufficiency Conditions (Look these up in Chaing.)

A. Concave Programming: the Kuhn Tucker Sufficiency Theorem (Chiang 21.4, see also La Fuenta Theorems 1.14, 1.18, 2.1, and 2.2).

B. Quasi-Concave Programming: the Arrow Enthoven Sufficiency Theorem (Chiang 21.5, La Fuenta Theorem 2.2).

V. Problem Set (Collected next week)

A. Use the Lagrangian and Substitution Methods to characterize the utility maximizing level of goods G and H in the case where

i. $U=HG$ where $W = pG + H$, iii. $U = HG$ where $H^2 + G^2 = 20$

ii. $U = H(G-2)^2$ where $12 = G + 2H$,iv. $U = v(G) + u(H)$ where $H + G = W$

B. Use the Kuhn-Tucker Technique to characterize the utility maximizing level of goods W and L when:

i. $U = L^{.5} + 1/(1+W)$ and $24 \geq W + L$ with $W, L \geq 0$

ii. Interpret your result. What kind of "good" is W ?

Next week: the implicit function theorem and more applications and extensions of constrained optimization.