

More on Comparative Statics

Time, and Uncertainty

Topics: Envelop Theorem, Present Discounted Value, Expected Value

Applications: theory of the firm, consumer choice, cost benefit analysis.

I. The Implicit Function and Envelop Theorem

A. Last time we demonstrated the mathematical roots of the implicit function theorem. As noted, the implicit function theorem is a very useful way of demonstrating how the *optimal value* of a *control variable* changes as parameters of the decision problem change.

B. The *Envelop Theorem* has a similar application. It is used to determine how the optimized value of the *objective function* varies as parameters of the decision problem change. (That is, if Q^* is the profit maximizing output of some good, the envelop function can be used to determine how the firm's *profit level*, Π^* , changes as the wage rates, etc. change.)

- i. Demonstration of the Envelop Theorem. Let $\Pi = \pi(Q, w, r)$ where Q is the control variable and w and r are parameters of the decision problem (if one thinks of Q as output, then w would be the wage rate and r the rental cost of capital).
- ii. Differentiating Π with respect to Q and setting the result equal to zero allows us to characterize the profit maximizing output as: $Q^* = q(w, r)$. Substituting Q^* back into the objective function Π allows the profit realized at Q^* to be characterized.
- iii. $\Pi^* = \pi(q(w,r), w, r)$ The change in the Π caused by a change in w is:

$$d\Pi^*/dw = (\delta\pi^*/\delta Q)(\delta Q/\delta w) + \delta\pi^*/\delta w$$

- iv. Since we are looking along the upper bound or "envelop" of the profit function, we know that $\delta\pi^*/\delta Q = 0$ at Q^* (This is how we characterized Q^* in the first place).
- v. Consequently: $d\Pi^*/dw = \delta\pi^*/\delta w$ (evaluated at Q^*)
- vi. Notice that "v." implies that the direction of change in the maximum value of an *objective function* caused by a change in a decision parameter (change in profit caused by a change in wage rates) can be determined even if one cannot sign the partial derivatives of the ideal levels of a choice parameter (here Q)!

II. Intertemporal Choice, Time Discounting and Present Values

A. DEF: Let V_t be the value of some asset or income flow " t " time periods from the present date. Let r be the interest rate per time period over this interval.

- i. The present value of V_t is $P(V_t) = V_t/(1+r)^t$
- ii. It is the amount that you could invest today at interest rate r that would yield exactly amount V_t after t years. (**Note** that interest rate r is entered into the formula as a fraction, e. g. $4\% = .04$)
- iii. More generally, the present value of a series of income flows (which may be positive or negative) over T years:

$$P = \sum_{t=0}^T (V_t/(1+r)^t) \quad \text{where the interest rate is } r \text{ percent per period.}$$

- iv. That is to say, **the present discounted value of any series of values is the sum of the individual present values of each element of the series.**

B. In cases where a constant value is received through time, e.g. $v_t = v_{t+1} = v$, the above formula can be reduced to:

$$P = v [((1+r)^T - 1)/r (1+r)^T]$$

- i. This formula has many uses in ordinary personal finance. For example the present value of a lottery prize of \$50,000/year for twenty years is $(50,000) [(1.05)^{20} - 1] / (.05 (1.05)^{20})] = (50,000)(12.4622) = \$623,110.52$ when the current interest rate is 5%/year.
- ii. The present value of \$50,000/year for twenty years is clearly much less than the \$1,000,000 that lottery sponsors often claim for such contests.
- iii. Another use of the present value formula is to determine one's monthly mortgage or car payments. This is just the reverse of a present value. Here you know P (the amount borrowed) and need to solve for " v " given the monthly interest rate r and the number of months over which the payments will be made, T .
- iv. What you are doing with a bank loan is providing the bank a cash flow approximately equal to the present value of the loan. (The bank profits by charging you somewhat more than its "own" market rate of interest.)

C. Intertemporal utility maximization problems generally express the relevant budget constraints in present discounted value terms.

- i. For example, suppose that an individual attempts to allocate current wealth in order maximize utility over two periods (the present and the future).
- ii. If $U = u(C_1, C_2)$ and $W = C_1 + C_2/(1+r)$, where C_1 and C_2 are consumption levels in the two periods, W is the present value of current wealth, and r is the relevant interest rate, either the Lagrangian or substitution methods may be used to characterize the optimal consumption expenditures in each year.
- iii. Note that first order condition(s) imply that the marginal rate of substitution between future and current consumption is equal to one plus the interest rate, $(1+r)$. This marginal rate of substitution between future and current consumption is sometimes called the *subjective rate of time discount*.

- iv. The implicit function theorem allows consumption in both periods to be characterized as a function of interest rates and wealth. The envelop theorem allows the effect of interest rates or wealth on maximal utility levels to be characterized.

III. Expected Utility and Decision Making under Uncertainty

A. Another common decision environment involves settings where some particular outcomes will occur in the near future, but the outcome can not be known beforehand. The simplest cases are coin tosses, rolling dice, and lotteries.

B. DEF: The mathematical expected value of a set of possible outcomes, 1, 2, ... N with values V_1, V_2, \dots, V_N and probabilities of occurrence P_1, P_2, \dots, P_N is

$$E(V) = \sum_{i=1}^N P_i V_i$$

- i. The expected value represents the long term *average* value of the distribution of possible outcomes.
- ii. A probability distribution has the properties: $\sum P_i = 1$ (something has to happen) and $P_i \geq 0$ for all i (where all "possibilities" have positive probabilities of occurrence and "impossibilities" have $P = 0$).

C. The *expected utility* associated with such a setting is calculated in a similar manner:

$$E(U(V)) = \sum_{i=1}^N P_i U(V_i)$$

where the value of the possible outcomes is now measured in utility terms.

D. Utility functions that can be used to calculate consistent expected utility values are called Von-Neumann Morgenstern utility functions.

- i. Von-Neuman Morgenstern utility functions are bounded and continuous.
- ii. Von-Neuman Morgenstern utility functions are also "unique" up to a linear transformation (and are considered by some to be a form of *cardinal* utility).

E. DEF: An individual is said to be *risk averse* if the expected utility of some gamble or risk is less than the utility generated at the expected value (mean) of the variable being evaluated.

- i. Note that this implies that any utility function that is strictly concave with respect to income, exhibits risk aversion. (Why?)
- ii. A *risk neutral* individual is one for whom the expected utility of a gamble (risky situation) and utility of the expected (mean) outcome are the same.
- iii. A risk preferring individual is one for whom the expected utility of a gamble is greater than the utility of the expected (mean) outcome.
- iv. The more risk averse a person is, the greater is the amount that he or she is willing to pay to avoid a risk.

F. More formally, the degree of risk aversion can be measured using the *Arrow-Pratt* measure of (absolute) risk aversion: $r(Y) = -U''(Y)/U'(Y)$

IV. Problems (collected next week)

A. Suppose that Acme is a profit maximizing monopolist facing the inverse demand curve $P = p(Q, Y)$ and production costs $C = c(Q, w, r)$ where Y is average consumer income, w , is the average wage rate of those employed by Acme, and r is the prevailing market rate of interest. (Assume that p and c have the conventional first and second derivatives.)

- i. Characterize Acme's profit maximizing output level.
- ii. Characterize how Acme's output will increase as household income increases.
- iii. Characterize how Acme's profits change as interest rates increase.
- iv. Characterize how Acme's profits change as household income increases.

B. Suppose that Al wins the lottery and will receive \$100,000/year for the next twenty five years.

- i. What is the present value of his winnings if the interest rate is 4%/year?
- ii. How much more would a prize that promised \$100,000/year forever be worth? (Hint: find the limit of formula IIB as T approaches infinity.)

C. Suppose that Al can purchase "replacement" insurance to eliminate the down side risk of fire in his home. Suppose that in its current state, the house is worth 200k and that after the fire it would be worth 100k. The probability of a fire is P . Replacement insurance increases the after-fire value of the house back to 200k.

- i. If Al is risk neutral, what is the highest price that he will pay for replacement insurance?
- ii. If Al is risk averse with $U = V^{.75}$, where V is the value of the house, what is the highest price that he would be willing to pay?

D. Suppose that Al can purchase lottery tickets for \$5.00 each and that the probability of winning the lottery is P . If Al wins, he will receive \$50,000 dollars per year for 20 years. The twenty year interest rate is 3%/year.

- i. What is the highest price that Al will pay for a ticket if he is risk neutral?
- ii. Determine how Al's willingness to pay for the ticket increases as P , the probability of winning, increases and as the interest rate diminishes.

Next Week: More on the intertemporal choice and decision making under uncertainty.