

## Some Implications of Optimizing Behavior

In the previous lectures, we have seen how the mathematics of constrained optimization can be used to analyze core neoclassical choice environments of the firm and consumer behavior; today we explore some other choice settings where individuals can be modelled as constrained optimizers. The core model of human behavior used in economic analysis of the consumer and firm can also be applied to analyze other economic, political, and sociological decision environments.

The notes show how the basic tools can be applied to analyze Time Allocation, Duopoly, and Imperfect Information. The lecture will explore these and other choice setting from the economics literature.

### I. Time Allocation

- A. The theory of time allocation is fundamentally similar to the theory of wealth allocation. It is a decision in a setting of scarcity. Just as an individual has only so much wealth, he only has so much time. A rational individual will allocate his or her time in the manner which maximizes his or her utility.
- B. A time allocation problem that is central to economics is the Labor - Leisure choice. Suppose that  $U = u(Y, L)$  where  $Y$  is labor income and  $L$  is leisure. Suppose that the individual earns  $w$  dollars an hour, and that hours spent working,  $W$ , plus hours of leisure,  $L$ , sum to 24 hours.
  - i. The objective function is the utility function, the constraints are  $Y = wW$  and  $W + L = 24$ .
  - ii. The constraints imply that income can be written as a function of leisure:  $Y = (24 - L)w$ . Consequently, the optimization problem can be rewritten as:  
 maximize  $U = u((24 - L)w, L)$
  - iii. Differentiating with respect to  $L$  characterizes the utility maximizing level of leisure:

$$U_Y(-w) + U_L = 0$$

- iv. The implicit function theorem implies that (iii) can be used to characterize the utility maximizing level of Leisure.  $L^* = l(w)$ . From the constraints we also know the implied labor supply is  $W^* = 24 - L^*$  which is similarly a function of wage rate  $w$ .
- v. The implicit function differentiation rule can be used to characterize how the demand for labor varies with wage rates.  

$$\frac{[-U_Y - w(24 - L)U_{YY} + (24 - L)U_{LY}] / -[w^2 U_{YY} - 2wU_{YL} + U_{LL}]}$$
- vi. The term inside the brackets of the denominator is negative if the first order conditions characterize the utility maximum. (why?) So, the sign of this partial derivative is determined by the numerator.
  - a. The first term of the numerator is unambiguously negative. The marginal utility of wage income is positive.

- b. The next term is unambiguously positive, because under the strict concavity assumption there is diminishing marginal utility of income. The last term may be positive or negative according to whether  $U_{LY}$  is positive or negative.
- c. If the last two terms are "small," leisure will decline as wage rates increase and labor supply  $(24 - L^*)$  will therefore increase as wages rise. In this case, the supply curve for labor is upward sloping.

### II. Cournot Duopoly

- A. The Cournot model of duopoly assumes that two firms sell their outputs in the same market at the same price. Both firms make their output decisions *independently* of one another in the sense that each assumes that the other's output will not change in response to their own decision. (This sort of "independence" is still the most common assumption in modern non-cooperative game theory.)
- B. Suppose that the inverse demand curve for "name brand cola" is  $P = a - bQ$  where  $Q$  is the total output of "brand name" producers, here Pepsi and Coke, and  $a$  and  $b$  are constants. Each firm's production costs rise with output, wages and the cost of capital,  $C = c(Q_i, w, r)$ .
  - i. Both Pepsi and Coke attempt to maximizing profit given the output of the other. The profit realized by Coke is  

$$\Pi = PQ_c - C_c = (a - b(Q_p + Q_c))Q_c - c(Q_c, w, r)$$
  - ii. Differentiating with respect to  $Q_c$  and setting the result equal to zero allows Coke's profit maximizing output to be characterized for a given output level of Pepsi ( $Q_p$ ).  

$$(a - b(Q_p + Q_c)) + (-b)Q_c - C_{Q_c} = 0 \quad (\text{e. g. find } Q_c^* \text{ s.t. } MR = MC)$$
  - iii. The implicit function theorem implies that one can characterize Coke's "reaction function" (or best reply function) as:  

$$Q_c^* = q_c(Q_p, a, b, w, r)$$
  - iv. This function shows that the profit maximizing output of Coke varies with the output level of Pepsi, given input prices. The implicit function differentiation rule allows the slope of this reaction function to be characterized. Applying this rule to equation (ii) yields:
  - v.  $dQ_c^*/dQ_p = [-b] / [-2b - C_{Q_c Q_c}] < 0$
  - vi. The reaction function of Coke is a *downward sloping* in Pepsi's output if the linear market demand curve is downward sloping, that is if  $b > 0$ , and the cost function exhibits increasing marginal costs at chosen output levels.
- C. A similar reaction curve can be found for Pepsi which characterizes Pepsi's output as a function of the output level of Coke and its input prices.  

$$Q_p^* = q_p(Q_c, a, b, w, r)$$
  - i. It will also be downward sloping given  $b > 0$  and production in a range of increasing marginal cost. (Why?)
- D. In equilibrium, both firms will be on their reaction functions (that is each will be producing at the level which maximizes their profits).
  - i. Mathematically the equilibrium condition is written as:

$$Q_c^* = q_c(Q_p^*, a, b, w, r) \text{ and} \\ Q_p^* = q_p(Q_c^*, a, b, w, r)$$

- ii. Geometrically this occurs at the intersection of the two reaction functions.
- iii. Profits and prices can be determined by substituting the equilibrium outputs into the original profit and demand functions.
- iv. Note that the equilibrium price is *also* an implicit function of demand characteristics  $a$  and  $b$ , and input prices  $w$  and  $r$ .

### III. Imperfect Information

- A. The role of information in economic decision making is often neglected in the core models of the firm and consumer behavior. Yet it is clear that these models implicitly assume that firms and consumers have a great deal of information. For example, consumers are assumed to know their own preferences and constraints. The latter implies that consumers know the prices of all of the goods over which they allocate their income. [The "law of one price" is often invoked to simplify the analysis. Under the law of one price, each consumer knows the best, and consequently only, price at which they may purchase goods.]
- B. As any one who has gone shopping can attest, the same good often trades at a variety of prices (ignoring locational attributes of the good). Because knowledge of the price distribution can be used to "save money," consumers will often invest in price information, if it seems worthwhile.
- C. The search model of information (Stigler, JPE 1961) shows that individuals will acquire price information up to the point where the marginal cost of additional information equals (or exceeds) the expected value of additional information.
- D. Suppose, in a two good model there is one good for which price information is costlessly available and another for which price information is costly. The utility function is  $U = u(X_1, X_2)$  and the constraints are  $W = P_1 X_1 + P_2 X_2 + P_I I$  and  $P_2 = P_2^e = f(I)$  where  $I$  is the amount of price information acquired and  $P_i$  is its cost per unit. The expected price of  $P_2$  declines as more information is acquired. (Why?)
  - i. Under either substitution or Lagrangian methods one can find that the expected utility maximizing investment in information requires:  $X_2 \frac{dP_2}{dI} = P_I$
  - ii. That is to say, information will be acquired up to the point where the *expected reduction* in expenditures on  $X_2$  equals the marginal cost of the information purchased. (Does the marginal utility of the good matter?)
  - iii. Beyond this point, consumers will rationally remain ignorant of the existence of better prices.
  - iv. The implicit function theorem implies that the optimal investment in information varies with the cost of information, the price of the good whose price is known and personal income.  $I^* = i(P_1, P_1, W)$

- v. Note that other factors, such as frequency of purchase, which might affect the shape of the expected price function do not explicitly appear in the information demand function, because they have not been explicitly modeled.
- vi. (Note: another concept of imperfect information that is sometimes confused with this optimal search/sample size conception is the notion of rational ignorance.)

### IV. Review Problems (Reviewed during midterm review)

- A. Suppose that Al is a "grade maximizing" student who is attempting to allocate her time between studying for two courses at the end of the semester. Her expected grade in either course increases as hours devoted to studying that course increase, but at a diminishing rate. She can only study for one course at a time. She has  $S$  hours to allocate.
  - i. Develop an objective function and constraints for analyzing Al's time allocation problem.
  - ii. Find the first order conditions that characterize her grade maximizing allocation of time across the courses.
  - iii. Use the implicit function theorem to generate a "studying supply" curve for one of the courses modelled. How does studying time for this course change as  $S$  changes?
- B. Suppose that  $N$  risk neutral competitors participate in a lottery game with a fixed prize. Each player may purchase as many tickets as he wishes. The prize is awarded to the player whose name is drawn from a barrel containing all of the tickets. So, the expected prize for player  $i$  is  $R [ t_i / (t_i + t_o) ]$ , where  $R$  is the value of the prize,  $t_i$  is the number of tickets purchased by player  $i$ , and  $t_o$  is the number purchased by all other players.
  - i. If tickets cost  $C$  dollars each, find the number of tickets that maximizes player 1's expected reward for a given purchase by all other players.
  - ii. Find a similar "best reply" function for some other player.
  - iii. An equilibrium may be said to arise in this game when no player has a reason to alter his ticket purchases.
  - iv. Find (or characterize) the equilibrium to this game when  $N=2$ , when  $N=3$ , when  $N=n$ .