

EC 812 -- L3A: Micro Foundations of Supply and Demand: Mathematical Appendix

VII. The Mathematics of the Neoclassical Firm's Supply Curve

A. Derivation of a firm's long run and short run cost curves

- i. A neoclassical firm (owner) is characterized by its technologically efficient production function: $Q = q(L, K, R)$
- ii. where L is the quantity of labor employed, K is the quantity of capital rented, and R is the quantity of raw materials used.
 - a. Productive inputs are often summarized as land, labor and capital.
 - b. Technology is implicitly represented by the shape of the production function. In cases where one is interested in modeling the effect of improved technology on a firm's output and hiring decisions, a parameter representing technology, say T, can be added to the variable list, e. g. $Q = q(L, K, R, T)$.
 - c. The production function can be generalized to include the possibility of heterogeneous inputs, say different kinds of labor or capital goods, by interpreting each of L, K, and R as vectors of their respective categories of inputs. Multiple outputs can likewise be modeled by considering Q to be a vector of output/products rather than a single product.
- iii. The production function is in effect an efficient recipe for producing output(s) from various inputs (ingredients).
- iv. Generally, the production function is assumed to exhibit diminishing marginal returns "in the range of interest."
 - a. This allows for an early range where increasing returns occurs, a middle range where constant returns occurs and, finally, "the range of interest" where diminishing returns occurs.
 - b. That is to say the production function does not really have to be concave or strictly concave over the entire range of interest, but only in the range where outputs will be chosen. Recall that the second order conditions only require checking at the values (Q^*) determined by the first order conditions.
- v. In the case where the firm hires inputs in a competitive market, its costs are simply the sum of the amounts paid for the inputs that might be used.

$$C = wL + rK + pR$$
- vi. The *only input combinations that a net income maximizing firm owner would intentionally use are those which minimize the cost of producing any given output, say Q^* .*
- vii. Mathematically this can be characterized using the Lagrangian method:

Minimize $C = wL + rK + pR$ subject to $q(L, K, R) = Q^*$.

 - a. Form the Lagrangian: $\mathcal{L} = wL + rK + pR + \lambda(q(L, K, R) - Q^*)$ and differentiate with respect to the four control variables L, K, R, and λ . This yields four first order conditions that will be simultaneously satisfied at the cost minimizing combination of inputs for producing output Q^* .
 - b. $w - \lambda Q_L = 0$, $r - \lambda Q_K = 0$, $p - \lambda Q_R = 0$, and $q(L, K, R) - Q^* = 0$
 - c. Note that any two of the first three FOC's can be manipulated to characterize the tangency condition between an isocost line and an isoquant, for example $w/r = Q_L/Q_K$.
 - d. That is to say, cost minimizing production methods equate the marginal rate of technological substitution between inputs and their relative price (ratio of marginal costs).

- e. The implicit function theorem allows all four endogenous variables to be characterized as functions of the parameters of the optimization problem: $L^* = l(Q^*, w, r, p)$, $K^* = k(Q^*, w, r, p)$, $R^* = r(Q^*, w, r, p)$, and $\lambda^* = \lambda(Q^*, w, r, p)$.

- viii. Substituting these functions, characterizing input usage, into the original cost function(s) defines the firm owner's *total cost function*. Here, the lowest cost for producing Q^* at given input prices is: $C^* = wL^* + rK^* + pR^*$
- ix. Usually this *long run cost function* is simply summarized as $C = c(Q, w, r, p)$
 - a. If production takes place via a process that exhibits constant returns to scale, note that doubling inputs will double outputs *and costs*. (Here the firm has to buy exactly twice as many inputs to produce twice as much output. Thus, output $2Q$ costs exactly twice as much as Q .)
 - b. If output Q costs C , $2Q$ costs $2C$ and average cost in each case is Q/C . Thus, a *constant returns* production function implies that the firm has a *horizontal average cost curve*.
 - c. If output increases at a decreasing rate, one will have to more than double inputs to double outputs, consequently production costs will more than double. That is to say, if production is subject to *diminishing marginal returns* then *production costs will rise at an increasing rate*. (Average variable costs will be rising over a range of diminishing returns.)

B. Derivation of a firm's Long Run supply function

- i. A firm's *supply function* characterizes its output as a function of output prices and other parameters of its optimization problem that affect its profits, chiefly input prices.
 - ii. If the firm is a *price taker*, it can sell as much as it wants to at a given price, P. So total revenue is $R = PQ$. Given its cost function $C = c(Q, w, r, p)$, derived as above, we can now *characterize its profit as*: $\Pi = PQ - c(Q, w, r, p)$
 - iii. Differentiating with respect to Q generates a single first order condition that characterizes the firm's *profit maximizing output, Q^** :
 - iv. $P - C_Q = 0$, or $P = C_Q$ e. g. produce the output that sets marginal cost equal to marginal revenue. (Here, MR = the market price of the firm's output.)
 - v. The implicit function theorem allows this output to be characterized as a function of the various parameters (exogenous variables) of the firm's optimization problem:

$$Q^* = q(P, w, r, p)$$
 - vi. This function, $Q^* = q(P, w, r, p)$, *is the firm's long run supply function*.
 - vii. Note also from (iv) above that the firm's long run supply function is the *inverse* of its long run marginal cost curve. [To see this geometrically, draw a diagram with the firm's long run marginal cost curve, vary price (marginal revenue) and plot price and the firm's profit maximizing output on a second graph.]
- C. The firm's *short run supply curve* is derived in the same manner, except that, now, it is assumed that *some potentially variable factor, usually capital, is fixed* for the time period of interest.
- i. Derive the firm's short run variable cost curve by holding some factor of production constant, say $K = K^0$. Again use the Lagrange method, but this time you do *not* differentiate with respect to the fixed factor.

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- ii. The firm's *variable cost* is the $V = wL^* + pR^*$, its total *fixed cost* is $F = rK^0$, so its *total cost* is consequently: $C = V + F$ or $C = wL^* + pR^* + rK^0$.
- iii. Its short run cost curve can thus be characterized as $C = c(Q, w, p, r, K^0)$
[Remember that in the short run, the fixed factor is a parameter of the optimization problem rather than control variable.]
- iv. The firm's short run *profit maximizing output is characterized using this short run cost function*. Now, $\Pi = PQ - c(Q, w, r, p, K^0)$.
- v. Differentiating the short run profit function with respect Q yields: $P - C_Q = 0$ which is very similar to that developed above for the long run case. The only difference is that now C_Q denotes short run marginal cost rather than long run marginal cost. Now the profit maximizing output equates marginal revenue (here P) and marginal *short run cost*.
- vi. The implicit function theorem allows the short run profit maximizing output characterized by (v) to be characterized as a function of the (exogenous) parameters of the optimization problem, here $Q^* = q(P, w, p, r, K^0)$.
 - a. Sometimes K^0 is *erroneously* dropped from the variable list of the short run supply curve because it directly implies "fixed" rather than variable costs.
 - b. However, since the level of K affects decisions to employ L and R , the level of K implicitly affects marginal costs and therefore a firm's short run output.
 - c. On the other hand, derivatives with respect to fixed factor K are relatively uninteresting in most short run analyses because K can not be varied within the period of interest.
 - d. [Of course, such derivatives are not entirely irrelevant even in the short run. One might be interested in the effects of a natural disaster which reduced the stock of fixed capital, or be making long run plans in a setting where there is some uncertainty about prices.]
 - e. *The firm's short run supply curve is the inverse of its short run marginal cost curve.*

VIII. The Neoclassical Firm's Demand for Inputs

- A. A profit maximizing firm will, obviously, actually employ the particular combinations of inputs that maximize profits. This can be characterized easily by writing profit

EXPLICITLY in terms of inputs utilized:

$$\Pi = Pq(L, K, R) - wL - rK - pR$$

- i. Differentiating with respect to input levels yields three first order conditions:
 $PQ_K - r = 0$, $PQ_L - w = 0$ and $PQ_R - p = 0$
- ii. Note that the first term of each is the marginal revenue product of the input (factor) of interest. *The profit maximizing level of all inputs will equate their marginal revenue product with the marginal cost of that input* (here a factor price).
- iii. Moreover, note that any pair of first order conditions can be used to generate the tangency condition for economically efficient production, as characterized above. For example, the first two imply that $Q_L/Q_K = w/r$, that is that the marginal rate of technological substitution between labor and capital will equal their relative prices. *Profit maximizing use of inputs implies economically efficient production.*

IX. Derivation of an Individual's Demand for Goods and Services

- A. An individual's demand for products sold in markets is represented as a utility maximizing choice. Every individual is characterized by a utility function and initial endowment.
- i. The market value of the initial endowment is the individual's wealth. (The initial endowment may be simply represented as money, or as inputs and final goods that can be traded (sold) in the market for desired final goods).
 - ii. The *budget constraint* implied by a given endowment $X_1^0, X_2^0, X_3^0 \dots X_N^0$ (of goods 1 through N) that can be sold or purchased at prices P_1, P_2, \dots, P_N is
$$\sum P_i X_i^0 = \sum P_i X_i$$
 - iii. Individuals maximize utility given their budget constraint/opportunity set.
 - iv. In cases where a large number of goods are relevant for the analysis, one can use the Lagrange method to characterize the optimal pattern of consumption for given prices, which can be used to derive the individual's demand functions for each good or service.
 - v. Here: $\mathcal{L} = u(X_1, X_2, \dots, X_N) - \lambda (\sum P_i X_i - \sum P_i X_i^0)$
 - vi. The first N first order conditions are of the form $U_{X_i} - \lambda P_i = 0$. The last one is the budget constraint, $\sum P_i X_i^0 - \sum P_i X_i = 0$
 - vii. The implicit function theorem allows all $N+1$ endogenous variables (the goods bought or sold plus the Lagrangian multiplier) to be characterized as functions of the parameters of the individual's choice problem, here prices and the original endowments.
$$X_i^* = x_i(P_1, P_2, \dots, P_N, X_1^0, X_2^0, \dots, X_N^0)$$
 - a. (With the simpler money wealth representations of the individual endowment, it would include prices and wealth, $X_i^* = x_i(P_1, P_2, \dots, W)$)
- B. In the two good cases that I often use for most illustrations, the *substitution method* yields a first order condition that is generally more useful for partial equilibrium analysis.
- i. Here $U = u(X_1, X_2)$ and $W = P_1 X_1 + P_2 X_2$ which implies that $U = u(X_1, W - X_1 P_1 / P_2)$.
 - ii. Differentiating with respect to X_1 yields: $U_{X_1} - U_{X_2} (P_1 / P_2) = 0$.
 - iii. Good X_1 will be purchased at the level, X_1^* , where its marginal subjective benefit equals its marginal subjective cost from reduced consumption of good X_2 .
 - iv. At this point, $X_2^* = (W - P_1 X_1^*) / P_2$ units of good X_2 are purchased.