

I. An Introduction to the Microeconomics of Public Economics.

- A. The first half of this handout provides a condensed overview of the basic geometry of public finance with some extensions to the theory of regulation and public choice.
- i. It is intended to be a review for students who have had undergraduate courses in public finance and regulation, and as a quick overview for those who have not.
 - a. We will draw upon this “tool bag” repeatedly during this course.
 - b. These tools provide much of the neoclassical foundation for public economics, both as it was being worked out in the first half of the twentieth century and as it was applied and extended in the second half.
- B. The second half of the handout reviews some of the core mathematical tools of public economics.
- i. Most of these results and approaches are taken for granted in published research.
 - ii. Other tools are now used, but most are grounded in those covered in this handout or are “ideosyncratic” ones, used only in a handful of papers.
- C. Part I begins with a somewhat novel net-benefit maximizing representation of rational choice.
- i. That model is used to derive supply and demand curves.
 - ii. It also provides a useful logical foundation for a variety of geometric, algebraic, and calculus representations of the burden of taxation.
 - iii. It can also be used for normative analysis, if we accept Pigou’s applied form of utilitarianism which economists tend to refer to as welfare economics.
 - iv. The same geometric approach can also be used to analyze (a) externality problems, and (b) public goods problems.
 - ◆ In future lectures, this approach will also be used to analyze electoral and interest group--based politics
- D. Part II develops a more or less parallel calculus-based analysis of taxes and public goods.

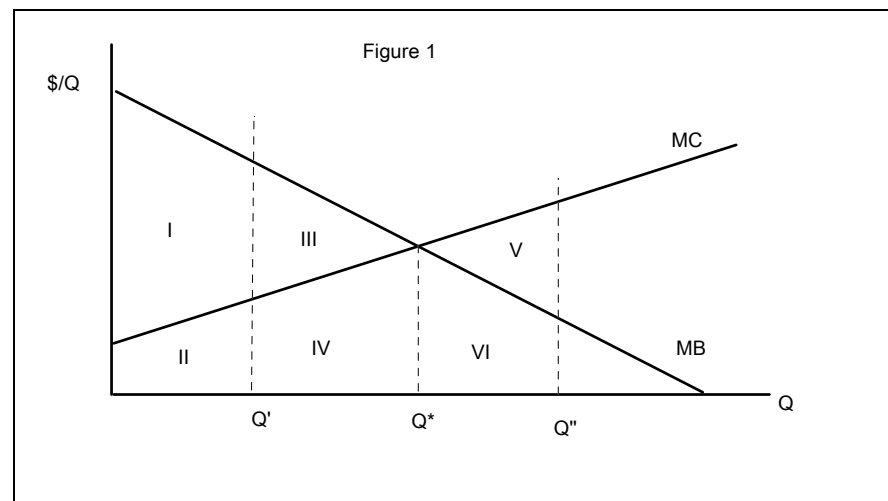
Part II: On the Geometry of Neoclassical Public Economics

I. Rationality as Net Benefit Maximizing Choice

- A. Nearly all economic models can be developed from a fairly simple model of rational decision making that **assume that individuals maximize their (expected) private net benefits.**
- i. Consumers maximize consumer surplus: the difference between what a thing is worth to them and what they have to pay for it.

$$CS(Q) = TB(Q) - TC(Q)$$
 - ii. Firms maximize their profit: the difference in what they receive in revenue from selling a product and its cost of production:

$$\Pi = TR(Q) - TC(Q)$$
- B. The change in benefits, costs, etc. with respect to quantity consumed or produced is generally called Marginal benefit, or Marginal cost.
- i. DEF: **Marginal "X"** is the change in Total "X" caused by a one unit change in quantity. It is the slope of the Total "X" curve. "X" ∈ {cost, benefit, profit, product, utility, revenue, etc.}
 - ii. *Important Geometric Property:* Total "X" can be calculated from a Marginal "X" curve by finding the area under the Marginal "X" curve over the range of interest (often from 0 to some quantity Q). This property allows us to determine consumer surplus and/or profit from a diagram of marginal cost and marginal revenue curves.



C. Examples:

- i. Given the marginal cost and marginal benefit curves in Figure 1, it is possible to calculate the total cost of Q' and the total benefit of Q' . These can be represented geometrically as areas under the curves of interest. $TC(Q') = II$; $TB(Q') = I + II$.
- ii. Similarly, one **can calculate the net benefits** by finding total benefit and total cost for the quantity or activity level of interest, and subtracting them. Thus the net benefit of output Q' is $TB(Q') - TC(Q') = [I + II] - [II] = I$.
- iii. Use Figure 1 to determine the areas that correspond to the total benefit, cost and net benefit at output Q^* and Q'' .
- iv. **Answers:**
 - a. $TB(Q^*) = I + II + III + IV$, $TC(Q^*) = II + IV$, $NB(Q^*) = I + III$
 - b. $TB(Q'') = I + II + III + IV + VI$, $TC(Q'') = II + IV + V + VI$, $NB(Q'') = I + III - V$

D. If one attempts to maximize net benefits, it turns out that generally he or she will want to consume or produce at the point where marginal cost equals marginal benefit (at least in cases where Q is very divisible).

- i. There is a nice geometric proof of this. (The example above, C, nearly proves this. Note that $NB(Q^*) > NB(Q')$ and $NB(Q^*) > NB(Q'')$.)
- ii. In the usual chase, a net-benefit maximizing decision maker chooses consumption levels (Q) such that their own marginal costs equal their own marginal benefits.
- iii. They do this not because they care about "margins" but because **this is how one maximizes net benefits** in most of the choice settings of interest to economists.
 - ♦ (Another common choice that maximizes net benefits is $Q^* = 0$. Why?)
- iv. This characterization of net benefit maximizing decisions is quite general, and can be used to model the behavior of both firms and consumers in a wide range of circumstances.

E. Notice that this approach yields sharper conclusions than the standard utility-based models, most of which are consistent with empirical work and economic intuition.

- ♦ Demand curves slope downward and supply curves upward, as shown below.
- i. However, it does not generate comparative statics as well.
 - ♦ To do this requires fleshing out the benefit and cost functions.

- ♦ It is partly for this reason that most economic models begin with utility and/or production functions, even though sharp conclusions are rare from those perspectives.
- ii. Net benefit maximizing models can be regarded as an operational counterpart to utility function based analysis.
 - ♦ Dollars unlike Utils are observable and measurable.
 - iii. For many purposes the net benefit maximizing framework and geometry is more tractable and more easily understood than its calculus based counterpart.
 - ♦ This makes it a very useful tool for classroom analysis, especially in undergraduate classes, but also in graduate classes..

F. The same geometry can be used to characterize ideal or optimal policies if the "maximize social net benefits" normative theory is adopted and "all" relevant costs and benefits can be computed

- ♦ This approach can be used to rationalize Benefit-Cost analysis
- ♦ It is also widely used in essentially any diagram that represents dead weight loss or excess burden in terms of a numeraire good such as dollars or Euros.
- ♦ Similarly, most partial equilibrium analyses of problems associated with taxation, monopoly, externalities, and public goods are undertaken with this basic geometry -- or at least can most easily be interpreted as such.

G. That each person maximizes their own net benefits does not imply that every person will agree about what the ideal level or output of a particular good or service might be, nor does it imply that all rational persons are selfish--although they may be self centered, e.g. focus on their own assessment of net benefits however calculated.

- i. Most individuals will have different marginal benefit or marginal cost curves, and so will differ about ideal service levels.
- ii. To the extent that these differences can be predicted, they can be used to model both private and political behavior:
 - a. (What types of persons will be most likely to lobby for subsidies for higher education?)
 - b. What types of persons will prefer progressive taxation to regressive taxation?
 - c. What industries will prefer a carbon tax to a corporate income tax?)
- iii. In cases in which an individual is not-selfish, benefits or costs that accrue to others will affect his or her own benefits.

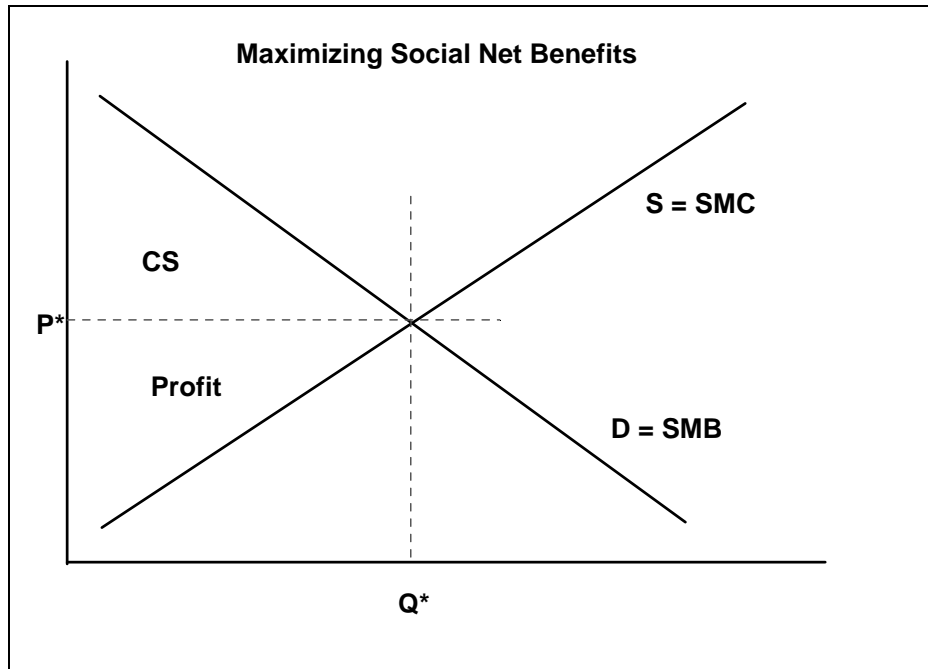
H. One can **use the consumer-surplus maximizing model to derive a consumer's demand curve** for any good or service (given their marginal benefit curves)

- i. This is done by: (a) choosing a price, (b) finding the implied marginal cost curve for a consumer, (c) use MC and MB to find the CS maximizing quantity of the good or service, (d) plot the price and the CS maximizing Q^* , and (e) repeat with other prices to trace out the individual's demand curve.
 - ii. This method of deriving demand implies that **demand curves always slope downward** because the demand curve is composed of a subset of the downward sloping sections of individual MB curves, namely all those that can maximize social net benefits for given prices.
 - ◆ If an individual's MB curve is monotonically downward sloping, then his or her demand curve is exactly the same as his or her MB curve.
 - ◆ If an individual's MB is not monotonically downward sloping (e.g. has bumps), then only a subset of the points on the MB curve will turn up on his or her demand curve (the ones that can characterize CS maximizing quantities).
 - ◆ In this last case, the individual's demand curve may be discontinuous, because there will be some quantities of a good that he or she will never purchase.
 - iii. To see this draw several MB curves and derive demand curves using the procedure (algorithm) above.
- I. Similarly, one can use a profit maximizing model (another measure of net benefit) to derive a competitive firm's **short run** supply curve, given its marginal cost curve.
- i. To do so, one (a) chooses a price (which is a price taking firm's MR curve), (b) finds the profit maximizing output, (c) plot P and Q^* , (d) repeat to trace out a supply curve.
 - ◆ This method of deriving a firm's supply curve implies that supply curves always slope upward because the supply curve is composed of a subset of the downward sloping sections of individual MB curves.
 - ◆ If a firm's MC curve is monotonically upward sloping, then its supply curve is exactly the same as its MC curve (where for SR supply one uses SR MC and for long run supply one uses the firms LR MC curve)..
 - ◆ If a firm's MC is not monotonically upward sloping (e.g. has bumps), then only a subset of the points on the MC curve will turn up on a firm's supply curve (the ones that can characterize profit maximizing quantities).
 - ii. In this last case, a firm's supply curve may be discontinuous, because there will be some quantities of a good that it will never produce.
 - iii. (Note that one does not need to use AVC curves to make his point.)
 - iv. To see this draw several MC curves and derive supply curves using the procedure (algorithm) above.

II. Markets and Social Net Benefits

- A. Market Demand can be determined by varying price of a single good and adding up the amounts that consumers want to buy of that good at each price.
- i. **Market Demand** curves for ordinary private goods, thus, can be shown to be "horizontal" sums of individual demand curves
 - ii. Similarly, **Market Supply** (for an industry with a fixed number of firms) can be derived by varying price and adding up the amounts that each firm in the industry is willing to sell at each price.
 - iii. **Market Supply** curves for ordinary private goods can be shown to be "horizontal" sums of individual firm supply curves.
 - ◆ In the short and medium run, the number of firms in the industry can be taken as fixed.
 - ◆ Long run supply varies according to whether one is in a Ricardian or Marshallian world (as developed further below).
- B. Note that derived in this way, it is clear that:
- i. Every market **demand curve is (approximately) the horizontal sum of the marginal benefit** curves of the individual consumers, because each consumer's demand curve is essentially his or her MB curve.
 - ii. Every short and middle run **market supply curve is (approximately) the horizontal sum of the marginal cost** curves of the individual firms in the market, because each firm's supply curve is essentially its MC curve.
 - iii. Consequently, market demand can be used as aggregate marginal benefit curves for consumers and supply curves as industry marginal cost curves for all firms in the industry.
 - a. Note that industry profit for any quantity can be calculated by using P^* as the marginal benefit (marginal revenue) curve for firms in the industry and the supply curve as industry marginal cost.
 - b. Similarly, market consumer surplus for any quantity can be calculated by using P^* as the consumer's MC curve.
 - ◆ It is often of interest to use the market clearing quantity and price for this consumer surplus and profit analysis, but this is not the only price and quantity combination of interest, as will be seen repeatedly in this course.
 - iv. Supply and demand curves can also be used to calculate "social surplus" or "social net benefits," and used in normative analysis.
 - a. The area under the demand curve between 0 and Q^* is the total benefit realized by all consumers in the market from consuming Q^* units of the good.

- b. The area under the supply curve between 0 and Q^* is the total cost of producing Q^* units of the good. (For short run supply curves, this will neglect fixed costs.)
- c. The difference between these two areas is the social net benefit (in dollars, euros, etc) realized through the market production, sales, and consumption of Q^* units of the good.
- d. Note that the sum of profits and CS in this case (where there are no externalities) adds up to social net benefits.



- v. This allows demand and supply curves (which can be estimated) to be used to estimate the net benefits realized by all firms and consumers in a market (or industry).
 - a. It also allows social net benefits to be estimated.
 - b. Note that the “maximize social net benefits” norm is thus an operational norm.
 - c. (Note also that all these derivations and conclusions can be done without reference to indifference curves or utility functions, given the assumption of net-benefit maximizing behavior.)

- C. In competitive markets, prices tend to move to "market clearing levels," that is to prices that set the total quantity supplied by all firms equal to the total amount demanded by consumers. (This defines equilibrium market P^* , and Q^* .)
 - i. In competitive markets, this occurs where the supply and demand curve cross.
 - ii. At any other price, there will either be surpluses (which tend to cause prices to fall) or shortages (which tend to cause prices to rise).
 - iii. Note that this is, in principle, an entirely decentralized process requiring governments to do nothing more than enforce property rights and contracts.
- D. In the absence of externalities or market concentration, **markets tend to produce social net benefit maximizing outputs!**
 - i. Note that the "market clearing" price causes markets to produce the **social net benefit maximizing** level of output (in cases where there are not externalities, e.g. relevant costs or benefits).
 - ♦ Q^* sets social marginal benefit (the demand curve) equal to social marginal cost (the supply curve).
 - ♦ This is one very widely used **normative** argument for using markets as a method of organizing the production of useful services.

E. APPENDIX on two alternative models of long run supply.

- i. In **Ricardian** models of supply, the same geometry and logic can be used for long run supply.
 - ♦ (Ricardian long run supply assumes that each potential supplier has a different LR MC curve. These vary among firms because farm fields and oil wells can be more or less costly to develop, and may be closer or further from transport centers. Their managerial talent may also vary.
 - ♦ Given the Ricardian assumption, the same geometric logic can be used to develop LR supply as in short run analysis.
 - ♦ The LR supply for Ricardian markets is again a horizontal sum of individual supply (and MC) curves, and can be used to approximate the industry's LR marginal cost curve.
- ii. In contrast LR supply in the **Marshallian** context occurs as identical firms (e.g. with identical MC and ATC curves) entry and exit the industry of interests.
 - ♦ Incentives for exit and entry end when profits fall to zero.
 - ♦ This implies that each firm is at the bottom of its LR ATC curves, which allows long run supply curves to be characterized as the industry's LR ATC rather than its LR MC.
 - ♦ (This property also allows the equilibrium numbers of firms to be calculated for Marshallian industries using long run average cost curves of firms. In the

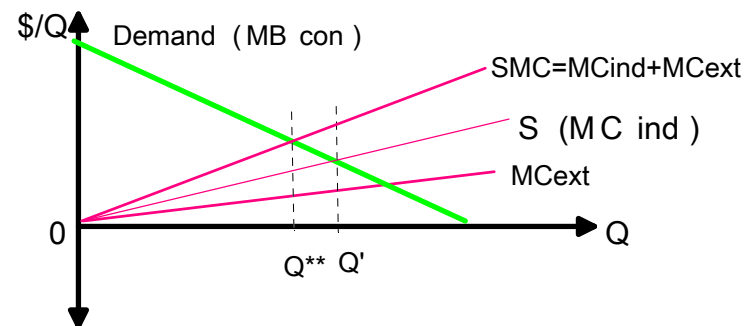
long run, efficient sized firms are simply replicated until market demand is satisfied.)

- iii. We will shift back and forth between the Ricardian and Marshallian long run perspectives according to the market at hand, but I tend to use the Ricardian perspective most often in class.

III. Markets, Externalities and Social Net Benefits

- A. In cases where **external costs** exist, however, market outcomes will (often) fail to maximize social net benefits, because (it is assumed that) external benefits and costs will not be fully accounted for in the calculations of economic agents.
- i. This is very easy to demonstrate within the net benefit maximizing framework.
 - ii. DEF: An externality may be said to exist whenever a decision made by an individual or group has effects on others not involved in the decision.
 - a. That is to say, an externality occurs whenever some activity imposes spill-over costs or benefits on persons not directly involved in the activity of interest.
 - b. If there are externalities, the market demand and supply curves (functions) will not include all marginal benefits or all marginal costs borne by persons in society.
 - iii. The missing benefits or costs can be represented with an external marginal benefit or external marginal cost curve.
 - a. To get social marginal benefits, one adds (vertically) the external marginal benefit curves to the consumer's marginal benefits (the demand curve).
 - b. To derive a social marginal cost curve, one adds (vertically) the external marginal cost curve to the industry marginal production cost schedule (the supply curve).
 - ♦ (Similarly, to derive a social marginal benefit curve, one adds (vertically) the external marginal benefit curve to the MB curve of consumers (the market demand curve).
 - ♦ (In most cases, there are assumed to be only external costs or benefits, not both--but clearly both are possible.)
 - c. The intersection of the social MB and MC curves characterizes the social net benefit maximizing production of the goods or services of interest.
 - d. **Note that with externalities, markets will no longer produce the social net benefit maximizing output of goods or services.**
 - ♦ In the diagram below, the market outcome Q' differs from Q^{**} , the social net benefit maximizing output.
 - ♦ In this case, $Q' > Q^{**}$, and output is larger than optimal from the perspective of the social net benefit maximizing norm.

Figure 3: The Excess Supply of a Polluting Activity



- B. The existence of externality problems, thus, provides a **normative** basis for government policy (if one wants to maximize social net benefits).
- i. In cases where significant external costs exist at the margin (at Q^*), markets will tend to **over produce** the output of interest relative to that which maximizes social net benefits.
 - ii. In cases where significant external benefits exist at the margin (at Q^*), markets will **under produce** the service of interest relative to that which maximizes social net benefits.
 - iii. Governments might adopt either regulations or taxes or a combination of the two discourage production in the first case (perhaps with Pigovian taxes) or encourage it (perhaps with Pigovian subsidies) in the second case.
 - ♦ (We will dig deeper into alternative remedies later in the course.)

IV. The Geometry of Public Finance: The Burden of Taxation

A. Taxation is the primary method by which governments finance themselves.

- i. Essentially all taxes transfer resources from the private to public sector, where government decision makers (both elected and un-elected) choose how the resources collected via taxation are allocated between providing services and redistribution.
- ii. Essentially all taxes shift resources to the government by threatening current resource holders (property owners, labor, international trading firms, etc.) with punishments of various sorts if they do not "give" their resources to the government's tax collectors.
 - a. In this sense, **all taxes are coercive at the point of collection.**
 - b. This contrasts with government bonds and ordinary fees for services, because such transactions are voluntary at the point of collection. Bond buyers and public service purchasers expect to be better off after the purchase, whereas tax payers normally feel worse off after paying the tax (although better off than had they not paid and been placed in jail).
- iii. On the other hand, whenever taxes are used to fund broadly desired services, taxation as a method of government finance can be regarded as voluntary in much the same sense that the amounts paid stores for their products can be regarded as voluntary.
 - ◆ In such cases, voters would rather "tax themselves" to pay for desired governmental services than go without those services.
- iv. There are a variety of non-tax sources of revenues, although these will not be given much attention in this tax review section.
 - a. For example, a good deal of government revenue comes from sales of bonds, e. g. borrowing.
 - b. To the extent that government borrowing is repaid, borrowing can be regarded as an implicit tax, because the government promises bond buyers that they will collect tax revenue in the future to pay interest and principal on the bonds.
 - ◆ The logic and politics of debt finance will be taken up later in the course.
 - c. There are sources of revenue that involve sales of government services or assets to the public.
 - ◆ Governments often charge tolls for bridges and highways and entry fees for parks and museums.
 - ◆ In addition governments occasionally sell assets such as building, mining permits, and land.
 - ◆
 - ◆

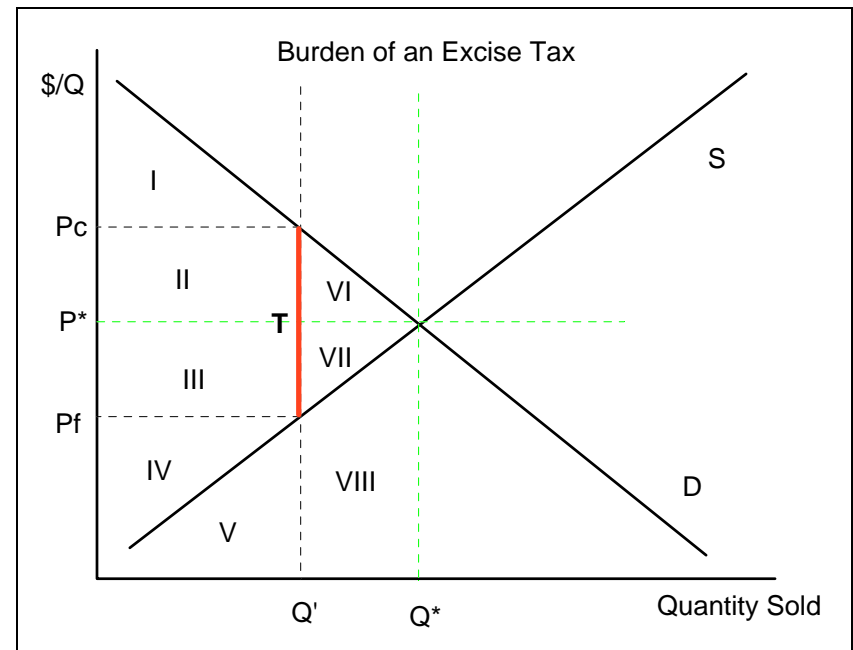
B. The burden of particular excise taxes can be measured in two ways:

- i. First, it can be calculated as a cash payment--in much the same way that payments for ordinary goods are calculated.
 - a. This is the most widely used measure by macro-economists, accountants, and newspaper reporters.
 - b. It is also occasionally implicitly used by public finance economists. When data on the slopes of relevant demand and supply curves are unavailable. In such cases, its often convenient as a first approximation to assume that supply is horizontal--as in a Marshallian competitive long run equilibrium--and so the full economic burden falls on consumers, as developed below.
- ii. Second, the burden of taxation can be calculated by determining the **losses** imposed on taxpayers as a consequence of the tax--that is to say the *opportunity cost of the tax*.
 - a. This is the approach used by most micro-economists (most of the time).
 - b. From this perspective, the burden of an excise or income tax can be measured as the **reduction of consumer surplus and profits** induced by the tax.
 - c. This approach measures how much worse off consumers and firms are because of the tax itself, which ignores any benefits they may receive from the tax financed services.
 - ◆ (In an indifference curve representation of tax burden, total burden is the change in utility associated with the tax.)
 - ◆ The benefit of a tax is the value of the services provided by the government using the tax revenues--but this tends to be neglected when talking about tax burden.
 - ◆ (Buchanan often suggests that "net burden" is the more relevant measure of the burden or benefit of a fiscal system.)
 - d. This measure of burden differs a bit from the money paid to the government in several ways.
 - ◆ First, the total burden of a tax is normally larger than the amount of money that taxpayers send into the treasury.
 - ◆ Most taxes have a **deadweight loss**. This can be measured as the extent to which "social surplus" is reduced by a particular tax, less the tax revenue generated.
 - ◆ Second, the distribution of the tax burden varies with market conditions (the slopes of the relevant supply and demand curves) rather than with who signs the check sent to the treasury.
 - ◆ Distributional effects are important politically, because they affect the net benefits associated with government programs for voters (and members of interest groups).

- ◆ Distributional effects are also important for most normative analyses of tax systems.
- iii. Because tax burden can be shifted in various ways, tax payments are often implicitly made by persons or firms who do not write checks to the treasury, and who may not be "obviously" affected by a particular tax law..
- a. For example, sales taxes are paid by firms in the sense that firms (or firm owners) actually write the checks deposited in the government's treasury.
- ◆ Thus, calculated as cash payments, one could say that the burden of a sales tax falls entirely on firms.
 - ◆ Alternatively it could be said that because sales taxes are tabulated separately on their receipts, consumers pay the tax.
 - ◆ The opportunity cost approach implies that the tax burden is likely to be shared by both firms and consumers.
- b. As a consequence of burden shifting and sharing, the persons most affected by a tax may not be the persons who "directly" pay the taxes by writing out a check to the treasury or IRS!

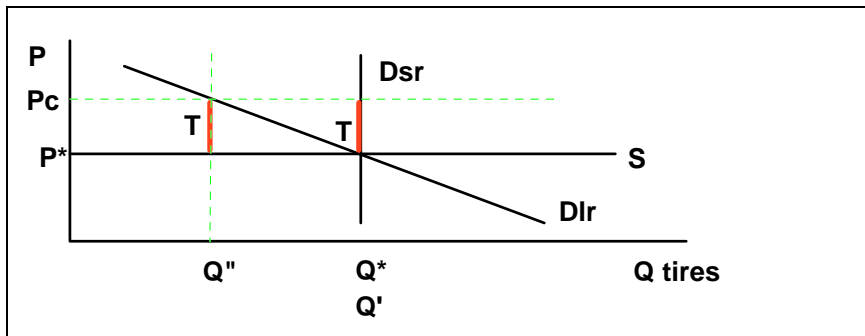
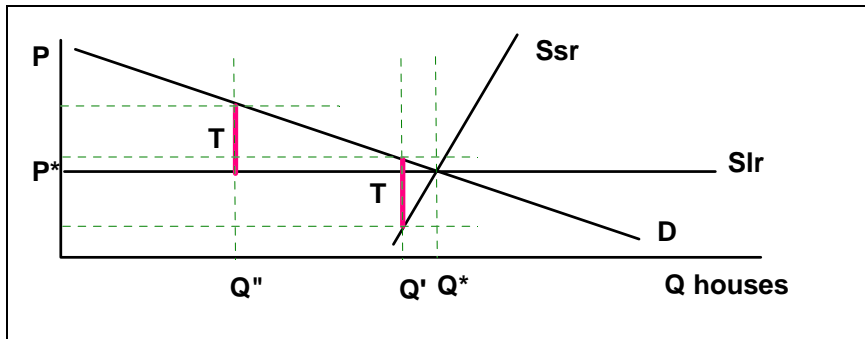
C. Illustration of the economic burden of an excise tax:

- i. Suppose that a market is initially in an equilibrium without taxes, so that demand equal supply at P^* . In this case, there is no "tax wedge" between the price paid by consumers, P_c , is the same as that received by firms, P_f ; so $P_f = P_c = P^*$.
- a. Now, suppose that an excise tax of T is imposed on each unit of the good sold in this market, as for example is done with tire sales in the US.
- b. After the tax is imposed, P^* is no longer the market clearing price:
- c. If T is simply added to P^* by firms, consumers will purchase too little at their new price ($P_c = P^* + T$) to match supply, which would remain at Q^* .
- d. On the other hand, if firms simply "ate" the tax, they would provide too little of the good to meet demand (at their after tax price of $P_f = P^* - T$). Supply would fall and demand would remain at Q^* if $P_c = P^*$ and $P_s = P^* - T$.
- e. To clear the market, thus, firms have to receive less than P^* per item sold, and consumers have to pay more than P^* .
- f. At the new equilibrium output, the demand curve will be exactly T dollars above the supply curve, and $Q_d(P_f + T) = Q_s(P_f)$.



- ii. This equilibrium output is shown in the diagram.
- a. At Q' , supply equals demand, if the price paid by consumers is exactly T dollars higher than the amount firms receive ($P_c = P_f + T$).
- ◆ Q' units of the good are sold, with $Q' < Q^*$.
- b. At this equilibrium, there is a sense in which the tax has simply been passed onto consumers, because $P_c = P_f + T$.
- ◆ However, there is another sense in which the **burden of taxation is shared** by firms and consumers, because both consumer surplus and profits have been diminished by the tax!
 - ◆ **Consumer Surplus falls** from area I + II + VI (before the tax at Q^*) to just area I after the tax is imposed and output falls to Q' .
 - ◆ Similarly, **Profit falls** from III + IV + VII (before the tax at Q^*) to area IV (after the tax at Q').
- c. The burden on consumers is II + VI, and that on firms is III + VII.
- iii. Note that **this distribution of the loss of consumer and firm net benefits occurs regardless of who actually writes the check to the state or federal treasury.**
- ◆ Price movements ultimately determine the actual division of burden between firms and consumers.

- ◆ If firms send in the check, their effective "payment" is reduced by the increase in price paid by consumers.
 - ◆ If consumers write out the checks, their effective "payment" is reduced by the price decrease absorbed by firms.
 - ◆ (Here it bears noting that in a Marshallian analysis, long run supply curves are always horizontal and so all the burden get shifted "forward" to consumers. This, however, is not the case in Ricardian markets. To see this, draw the case in which S is horizontal.)
- iv. The amount of revenue raised by the tax is $T \cdot Q'$.
- a. Q' units are sold and each pays a tax of T dollars.
 - b. The total tax revenue, TQ' , can be represented in the diagram area II + III in the diagram.
 - ◆ (Note that II + III is the area of a rectangle T tall and Q' wide.)
 - v. Notice that **the tax revenue is smaller than the "surplus" lost by taxpayers** (firms and consumers in the affected market).
 - a. The reduced profit plus the reduced consumer surplus equals $\{II + VI\} + \{III + VII\}$.
 - b. The total burden of this tax is VI + VII larger than the tax revenue.
 - c. This area of "excess burden" is sometimes referred to as the **deadweight loss of an excise tax**.
- D. Both the extent of the deadweight loss and the distribution of the tax burden vary with the slopes of the supply and demand curves.
- i. Generally, more of the burden falls on the side of the market with the least price sensitive curves.
 - a. If the demand curve is less elastic than the supply curve, more of the burden falls on consumers than on firms.
 - ◆ In the extreme case in which market demand is completely inelastic or the industry supply curve is completely elastic, all of the burden falls on consumers!
 - b. On the other hand if the demand curve is very elastic, because good substitutes exist, or the supply curve is relatively inelastic then more of the burden tends to fall on the firm.
 - ◆ In the extreme case in which the market supply of the product of interest is completely inelastic or consumer demand is perfectly elastic, all of the burden falls on suppliers.
 - ii. The excess burden of a tax tends to increase with the price sensitivity (slope or elasticity) of the demand and supply curves.
- E. Both supply and demand tend to be **more elastic in the long run than in the short run**, because more factors of consumption and production can be varied, consequently, the excess burden of taxation tends to be larger in the long run than in the short run.
- i. In cases in which long run and short run demand are the same, the fact that long run supply is relatively more price sensitive (elastic) than short run supply implies that the burden of a new tax or increase in tax tends to be gradually shifted from firms to consumer in the long run.
 - ◆ Marshallian competitive markets have perfectly elastic supply curves in the long run, which implies that narrow taxes on such products are shifted entirely to consumers in the long run.
 - ◆ Goods taxed at state and local levels that are sold in national and international markets also tend to have horizontal supply curves in the local or state markets (even in Ricardian markets).
 - ii. There are many cases in which consumer demand and industry supply is more price elastic in the long run than in the short run.
 - ◆ For example, consumer demand also partly depends on complementary capital goods like automobile, that can be varied in the long run as taxes change.
 - ◆ Europeans, for example, pay \$5.00 per gallon gasoline taxes and so, naturally, tend to drive small cars.
 - ◆ In such cases, a tax such as a gasoline tax may be gradually shifted from consumers to firms (owners of capital and natural resources) in the long run.
 - ◆ The reverse tends to happen in cases when firms make LR adjustments that they cannot make in the short run.
 - iii. In cases where both sides of the market (firms and consumers) are more price elastic in the long run than in the short run, the shift of burden will reflect their relative ability to adjust.
 - iv. All such long run adjustments imply that **deadweight losses** to narrow taxes, such as an excise tax, are **larger in the long run than in the short run**.
 - v. Illustrations: effects of an excise tax in the short run and long run for different kinds of markets



- a. Note that in the first case, supply is more elastic in the long run than in the short run, so the initial effect of the tax is largely on firms, but in the long run the burden is shifted to consumers.
 - ◆ The after tax price falls at first for firms, but rises back to P^* .
 - ◆ The price to consumers rises just a bit at first, but rises to $P^* + P_c$ in the long run.
- b. The second case is the case where demand is more price sensitive (elastic) in the long run than in the short run, but because supply is completely elastic in both the long and short run, the burden falls entirely on consumers in both the short and long run.
- c. *As an exercise, construct a case in which the burden falls entirely on firms in both the long and short run.*
- vi. In some cases, losses of consumer surplus may occur in other markets as a result of excise taxes.
 - ◆ For example high gasoline taxes encourage bicycle purchases, which tend to increase the price paid for bicycles, at least in the short run.
 - ◆ However, profits rise by nearly as much as the consumer surplus falls, so we will ignore these secondary effects in most of our analysis of tax burdens in this class.

V. A Review of Elementary Normative Principles of Taxation

- A. The ideas summarized in these diagrams are have often been used to characterize "normative theories" of taxation. (We will review these in more detail later in the course.)
- B. For example, one normative theory of taxation was proposed by Frank **Ramsay** in 1927. He argued that a system of excise taxes should attempt to **minimize the total excess burden** of the tax system.
 - i. A **Ramsay tax** system thus imposes higher taxes on markets with relatively inelastic supply and demand curves, and relatively lower taxes on markets with relatively large price sensitivities.
 - ii. If markets with perfectly inelastic demand or supply curves exist, government services can be financed without any deadweight loss at all, if taxes on such goods can generate sufficient revenues.
 - ◆ (Remember that taxes on products with inelastic supply or demand curves generate no deadweight losses.)
 - iii. A special case of such a tax is a tax on land--which is sometimes called a Georgist tax after **Henry George** who proposed financing government entirely with land taxes. The supply of land, after all, is perfectly inelastic (ignoring dikes and dumps).
 - ◆ (Analyze the limitations, if any, of a Georgist land tax.
 - ◆ Where does the value of a piece of land come from? Would there be allocative affects across different types of land? Would a Georgist land tax be neutral even if it is a Ramsay tax?)
- C. It has also been argued that a tax system should not directly affect *relative prices* across markets (see the indifference curve analysis in the appendix).
 - i. That is to say, a tax system should be **NEUTRAL**.
 - a. A perfectly neutral tax system would not affect private sector decisions across markets for private goods and services, because it would not affect relative prices faced by firms or consumers (although it does, of course, produces revenues for the government).
 - b. In this case, a government that tried to finance itself via a system of excise taxes would impose excise taxes so that prices increased by the same proportion in every market taxed.
 - c. Alternatively, the government could look for somewhat narrower tax sources that do not have relative price effects, such as a **lump sum** or **head tax**.
 - d. (The geometry of lump sum taxes and other nearly neutral taxes can be illustrated a bit more easily using indifference curves and budget constraints, as is done in the next section.)

- ◆ (Nonetheless, systems of excise taxes that generate a proportionate increase in all prices faced by consumers can clearly be illustrated with the tools developed above using demand and supply curves for several markets.)
- ◆ (Given the possibility of international emigration, can their actually be a tax that has no dead weight loss? **Discuss.**)

D. Another normative theory of taxation argues that one cannot determine the proper division of the tax burden without thinking about the services that will be provided.

- i. For example, Lindahl argues in favor of a **benefit tax**, that is a tax that imposes the greatest burden on those who receive the most valued services should pay the highest taxes.
 - ◆ Under an ideal Lindahl tax system, each person's *marginal tax rate would be set equal to the marginal benefits* she or he receives from government services.
 - ◆ (We will review Lindahl taxes in more detail later in the course.)
 - ◆ James Buchanan (who won a Nobel Prize in economics in 1986, partly for his contributions to public finance) tends to agree with Lindahl.
- ii. Buchanan argues that proper accounts of tax burden--should focus on net tax burden--that is, they should take account of the services financed by taxes as well as the taxes paid.
 - ◆ For example, if a person receives an especially valuable service from the government, it is possible that his or her "true" net tax burden is negative. Others who receive no services of value, might have positive net tax burdens.
 - ◆ **Ideally, all citizens would bear "negative" tax burdens in the sense that each person should receive services that are considered to be more valuable than the taxes paid.**
 - ◆ (We analyze the demand and supply of public goods and other government services below.)

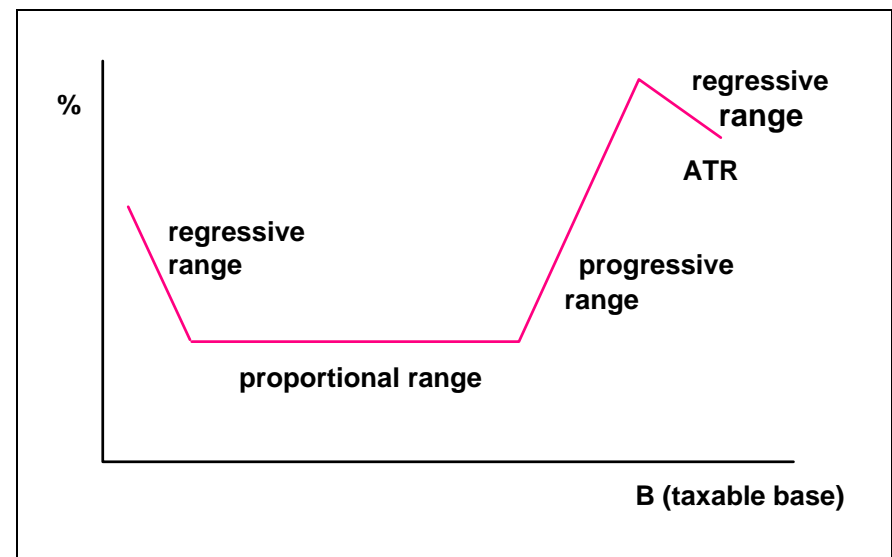
E. Other normative principles of taxation come are rooted in social norms and political philosophies (ideologies) of various kinds. These often focus attention on the **fairness** (or equity) of a tax system.

- i. For example, many argue that persons should pay based on their "ability to pay."
 - a. This notion of **fairness** tends to imply progressive income taxes.
 - b. For example, a "fair tax" might be one that caused all taxpayers should all sacrifice approximately the same "utility" (rather than net benefits) when they pay their taxes.
 - ◆ (Since the marginal utility of money tends to be smaller for rich persons than poor persons, more money would be collected from rich persons than from poor persons.)

- ◆ (That is, taxes should be progressive rather than regressive)
- ii. Others suggest that fairness requires all persons to pay be treated the same way under a tax system.
 - ◆ This notion of fairness tends to imply a flat tax--a proportional tax on income. (See Buchanan and Congleton 1998.)
 - iii. Others argue that all similar people be taxed in the same way (horizontal equity).

A. Definitions:

- a. A **progressive tax** is a tax whose average burden increases as the taxable base owned by an individual increases. [Such taxes often have marginal tax rates that increase with the base (increase with income), although not all progressive taxes have this property. Most income tax systems in industrialized countries are somewhat progressive.]
- b. A **proportional tax** is a tax whose average tax burden does not change with income. (Such taxes normally have a constant marginal tax rate, as true of most sales taxes and some income taxes. A flat (proportional) tax on income has the form: $T = tY$.)
- c. A **regressive tax** is a tax whose average tax burden falls with income. Such taxes often have declining marginal tax rates with ownership of the taxable base, however, not all regressive taxes have this property. An example of a regressive tax in the US is the social security tax--which has a cap on taxable income.
- d. (Note that these definitions simply characterize the tax code, rather than address redistributive questions as often done, especially in the media.)



B. Definitions and Relationships:

- ◆ The **tax base**, B , is that which is taxed (taxable income, sales of final goods and services, profits, property, gasoline, etc.).
 - ◆ The **average tax rate** of a particular tax often varies with an individual's holding of the taxable base. If an individual pays tax T_i on a holding of B_i , his average tax rate is T_i/B_i . (If $T_i = \$50$ and $B_i = 200$, the average tax rate for this tax is $50/200 = 0.25$ or 25%.)
 - ◆ The **marginal tax rate** of a particular tax is the change in taxes owed for a one unit increase in holdings of the taxable base, DT/DB . (So, if a tax payer earning 50,000/year pays a tax of 10,000 and a taxpayer earning 50001 pays a tax of 10,000.50, his or marginal tax rate is $0.50/1 = 50\%$. Fifty percent of each additional dollar earned is taken from the "last" dollar of income earned by a taxpayer earning 50,000/year.)
- a. Diagramming average and marginal tax schedules:
- ◆ If MTR is above ATR, then that ATR curve will be rising (the marginal tax rate will be pulling the average up).
 - ◆ If MTR is below ATR, then the ATR curve will be falling (the marginal tax rate will be pulling the average down).
 - ◆ -If the $MTR = ATR$, the ATR will be neither rising nor falling.
- b. Since individual decisions are determined by marginal cost and marginal benefits at various quantities, **it is the marginal tax rate rather than the average tax that directly affects tax payer behavior** in most cases.
- ◆ (Thus, one argument in favor of proportional, or indeed, regressive taxes, is that they may have smaller effects on economic activities than a revenue equivalent progressive tax.)
- A. (Peckman's estimates of the effective average and marginal tax rates faced by a typical American tax payer, often look a bit like this odd tax schedule.)
- i. (As an exercise try to determine what the marginal tax schedule that corresponds to this average tax schedule looks like.)
 - ii. (Explain briefly why Peckman finds regressive ranges of taxation at both the highest and lowest ranges of income.)
- B. Other normative tax theories are unconcerned with the fairness of the tax system. They argue that a tax system should attempt to promote economic growth--or at least minimize the reduction in growth associated with raising a given amount of revenue.
- i. Such persons often favor consumption taxes in order to encourage saving and investment.

- ii. The effects of a consumption tax on investment is easiest to illustrate with indifference curves and budget constraints, but the intuition behind the effect is simply based on supply and demand.
 - a. If the price of saving falls relative to consumption, individuals will consume less and save more.
 - b. And if savings increase, capital will be more rapidly accumulated, which leads to higher income levels and growth rates.)
 - c. The internationalization of capital markets weakens this rationale for consumption taxes, although VATs are widely used in Europe and sales taxes are widely used as source of state government finance.

VI. Analysis of the Effects of Taxes on Individuals Using Indifference Curves

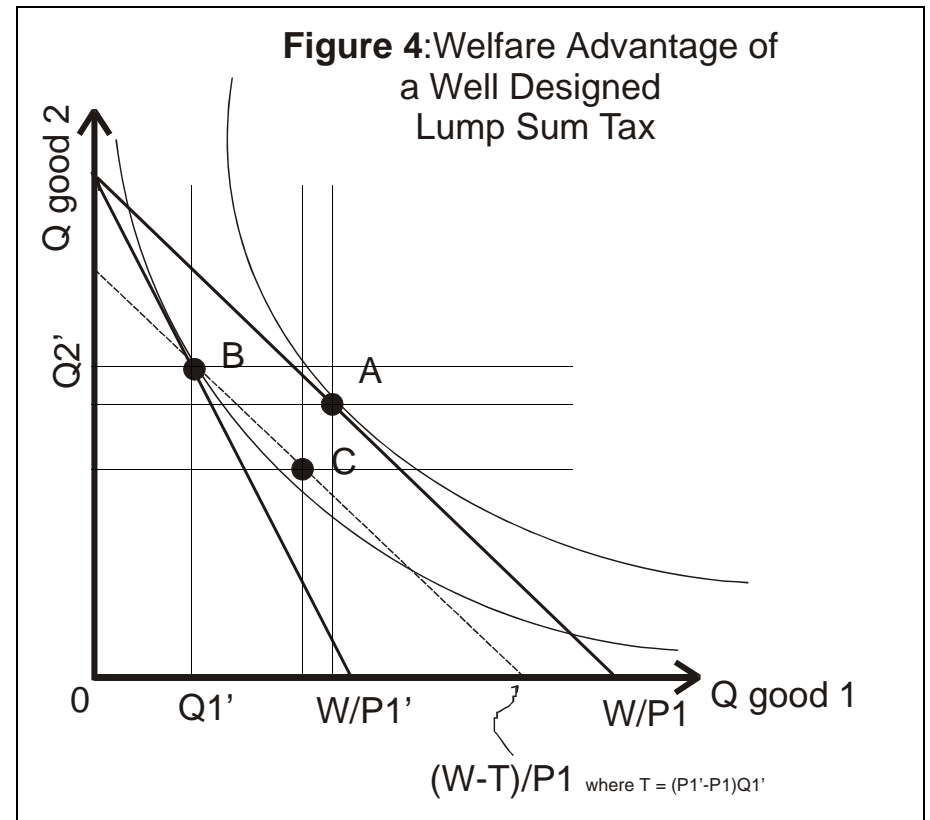
- A. The behavior effects of an excise tax can also be analyzed with indifference curves and budget constraints.
- B. Suppose that there are two goods, Q and X, both of which the consumer normally uses.
- i. Our previous analysis implies that the effect of an excise tax on a typical consumer is to raise the price of the taxed good from P^* to P_c .
 - ii. This increase in price affects the location of each consumer's budget set.
 - iii. It rotates the budget constraint from the untaxed end of the budget constraint and generates a new budget constraint that lies inside the original one at all points where the consumer purchases positive quantities of the taxed good.
 - iv. Suppose that "A" is the original bundle consumed by this consumer.
 - a. In this drawing the tax has increased price of good 1 from P_1 to P_1' (this price effect is taken from a supply and demand diagram)
 - ◆ In the case drawn, the new higher price causes the consumer to purchase bundle **B** instead of **A**. (Indeed, A is no longer feasible.)
 - b. If instead of an excise tax on good 1 a lump sum tax (or wealth or sales tax) had been used, the budget constraint would have shifted toward the axis, but the new budget constraint would have the same slope as the original one.
 - ◆ The "revenue equivalent" lump sum tax passes through point **B** and is parallel to the original (pre tax) budget constraint.
 - ◆ Note that a lump sum tax, would have allowed the individual to purchase a bundle like **C** which is on a higher indifference curve (not drawn) than bundle **B**.
 - ◆ This loss in utility (from being on a lower indifference curve) is another measure of the excess burden of a non-neutral tax on consumers.

- ◆ (Here it bears noting that this result requires a good deal of information about individuals to implement. It is not so easy to create these Pareto superior shifts in the tax base. [Explain why.](#))
- c. Much of the deadweight loss is a consequence of reduction in purchases of the taxed good, particularly that part which was generated by the "relative price" effect of the excise tax.
 - ◆ You learned in micro economics that every price increase has both a (relative price) substitution effect and a wealth effect on purchases of the good whose price has increased.
 - ◆ An excise tax that affects consumer prices has both a (relative price) substitution effect and an income effect on purchases of the good whose price has increased because of a tax.
 - ◆ A lump sum tax only has an income effect.
- d. Broad based taxes that do not effect relative prices have similar effects, and this diagram provides the basis for economic (normative) support for neutrality and for broad based taxes.

- C. The behavioral effect of a general tax and a lump sum tax tends to be smaller than that of an excise tax, because these taxes have only a wealth effect.
- i. A revenue neutral lump sum tax, a (neutral) general sales tax, and an income tax all shift each consumer's budget constraint towards the origin, but **these taxes do not affect the slope** of the consumer's budget constraint.
 - ii. Consequently, general taxes and lump sum taxes tend to have a smaller effect on behavior than excise taxes that raise the same amount of revenue. (There is no "substitution effect.")

D. The Algebra of Budget Constraints used for Tax Analysis.

- i. The slope of the budget lines can be calculated *for the lump sum, sales and income taxes.*
 - a. Recall that slope is "rise over run."
 - b. In the case without taxes, the slope of the budget line is $-(W/P_2) / (W/P_1)$, which simplifies to $-P_1/P_2$.
 - c. In the case of a lump sum tax, the endpoints of the new budget line are $(W-T)/P_1$ and $(W-T)/P_2$.
 - ◆ The slope of the new budget constraint is $-[(W-T)/P_2] / [(W-T)/P_1]$ which equals $-P_1/P_2$. ([Show this algebraically.](#))



- d. In the case of a an income tax, where W is treated as income, the after tax income is $(1-t)W$, so the endpoints of the new budget line are $((1-t)W)/P_1$ and $((1-t)W)/P_2$.
 - ◆ The slope of the new budget line is: $-[(1-t)W]/P_2 / [((1-t)W)/P_1] = -P_1/P_2$.
- e. In the case of a general sales tax the new after tax prices will be approximately $(1+t)P_1$ and $(1+t)P_2$. ([What assumptions about supply and demand are sufficient for this to be exactly true?](#))
 - ◆ The slope of the new budget line will be $-[(W)/(1+t)P_2] / [((1-t)W)/(1+t)P_1]$, which again can be shown to equal $-P_1/P_2$.
- f. All three of these taxes are "neutral" with respect to the choice illustrated in our diagram.
 - ◆ None of these taxes change the relative prices of goods 1 and 2. It remains $-P_1/P_2$ in each case.
 - ◆ (Note however that sales taxes have an effect on a consumers decision to save, and income taxes have an effect on a consumer's decision to work.)

E. In some cases, however, the purposes of a tax may be to change behavior.

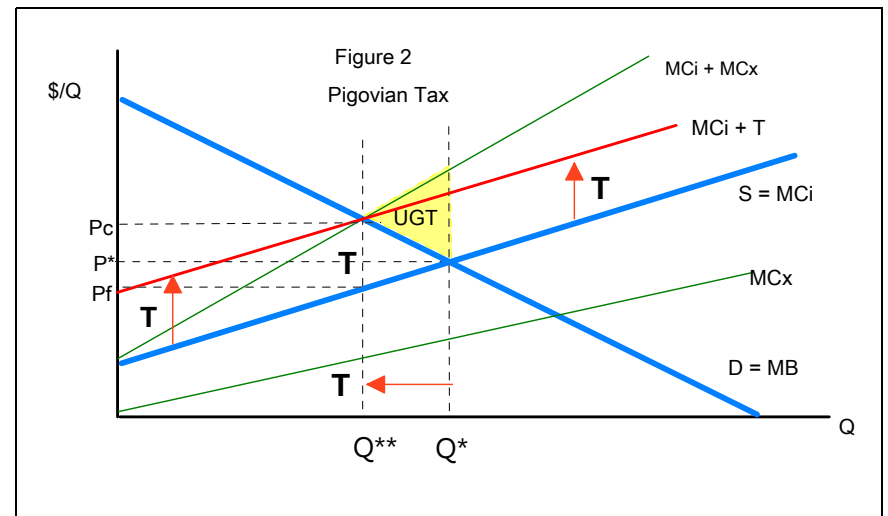
- i. In such cases, excise taxes and other "marginal" taxes will be more effective at altering behavior than lump sum or general taxes.
- ii. This is the case for Pigovian taxes used to "internalize" externalities and solve externality problems.
- iii. An interesting property of a Pigovian tax is that they have no deadweight loss in the usual partial equilibrium sense.
 - ◆ One can use the logic of the supply and demand with externality diagrams above to show this.
 - ◆ A Pigovian tax is set equal to the marginal external cost at Q^{**} (the Pareto optimal / Social Net Benefit maximizing output).
 - ◆ Given this tax, the market participants will shift their consumption and production patterns so that only Q^{**} is produced and sold.
 - ◆ Assuming that the tax revenue raised is spent in a reasonable way, the result will be an increase in social surplus rather than a decrease.
 - ◆ (Draw such a case and think about how social net benefits are affected by a Pigovian tax.).

VII. The Double Dividend: On Pigovian Taxation

- A. There are several possible collective management solutions to externality problems.
 - i. Elinor Ostrom won the Nobel prize in 2009 for her analysis of the great variety of such solutions (and related commons problems).
 - ii. We examine just a few of the classic economic solutions in this class--namely the ones most studied by economists.
- B. **Pigovian Taxes:** Excise taxes as a means of "internalizing" negative externalities
 - i. A Pigovian tax attempts to change incentives at the margin by imposing a tax (or subsidy) on the activity that generates the externality.
 - a. Notice that if the externality producer is subject to a tax equal to the marginal external cost (benefit) at the Pareto efficient level, the externality producer will now choose to produce the Pareto efficient output/effluent levels.
 - b. Such a tax (or subsidy) is said to **internalize** the externality, because it makes the externality producer bear the full cost of his actions (at Q^{**}).
 - ii. In principle, Pigovian tax schedules can have a variety of shapes, but for the purposes of this class we will assume that they are all "flat excise taxes" that impose the same tax on every unit of the product (or emission) produced.

- iii. Pigovian taxes may yield substantial revenues although this is not their main purpose.
 - a. Unlike a neutral tax, the main purpose of a Pigovian tax is to change behavior.
 - b. Unlike an ordinary excise tax, a Pigovian excise tax generates no excess burden (as developed below and in class.)
- iv. **Illustration of the Pigovian Tax**

- a. From our analysis of externalities, we know that market equilibria may not maximize social net benefits or necessarily realize all potential gains to trade.
 - a. These unrealized social net benefits (or gains to trade) are the triangle labelled UGT at Q^* in the diagram.
 - a. A Coasian bargain might be able to realize those social net benefits if transactions costs are low enough, but they can also be realized by adopting a tax that internalizes the externality.
 - b. Such a tax induces market participants to take account of the external marginal costs.



- a. The external cost at Q^{**} is the vertical distance from MC to the $MC + MCx$ curve.
- b. This distance is the level of an ideal **Pigovian tax**. If it is placed on production or sales of this product, it will internalize the externality.
- c. This ideal tax is labeled "T" in the diagram above.
- d. If a tax of T dollars per unit is imposed on the firm's output (or emissions) the firm will now face a marginal cost for production equal to $MC + T$.
- e. Given this new MC curve (which includes the tax that "internalizes" the externality) the firm will produce an output of Q^{**} , the Pareto Efficient level.

- i. Pigovian taxes can be a low cost method of solving an externality problem, because firms and consumers can all independently adjust to the tax.
 - a. However, the tax burden required to achieve the desired level of the externality generating activity can be very large, which can make both consumers and firms in the taxed industry worse off.
 - b. This tends to make Pigovian taxes politically unpopular (explain why).
 - c. (This cost can be reduced by using the revenues for desired public services or by rebating the revenues as lump sum subsidies to people in the market being taxed.)
- ii. Imposing a Pigovian tax requires that the marginal external damages be estimated.
 - a. This may be possible at Q^* , the output actually produced in the unregulated setting.
 - b. However this will be more difficult to do at Q^{**} because Q^{**} is not observed and has to be estimated using estimates of SMC and SMB.

C. On the Double Dividend

- i. **Note that unlike other excise taxes, a Pigovian tax increases social net benefits (assuming the tax receipts are not wasted).**
- ii. **The externality generating tax is over provided before the tax and is provided at SNB maximizing levels after the tax.**
- iii. **Thus, from a Ramsay perspective, Pigovian taxes should be used to provide as much revenue as possible.**
- iv. **(The only rival to Pigovian taxes are taxes on unfinished land, “Georgist” taxes.)**

D. **Pigovian Subsidies** are essentially similar to that of the Pigovian tax, except in this case the externality generating activity is under produced, and the subsidy attempts to encourage additional production.

- ♦ (Internalizing the externality in this case requires producers to take account of unnoticed benefits falling on others outside the decision of interest.)
- ♦ A Pigovian subsidy is set equal to M_{bx} at Q^{**} and will cause the market to produce Q^{**} units of the good after it is imposed.
- ♦ A Pigovian subsidy increases social net benefits (beyond the cost of the subsidy) and so has no DWL.

E. **The Coase Theorems: Coasian Contracts** as solutions to Externality and public goods problems.

- i. Privatization, per se, does not solve all externality problems, only those in which property rights are undefined or poorly defined in the original setting.

- ii. However, given both clear property rights and the ability to negotiation over externalities a subset of externality problems can be solved through bargaining and contract.
- iii. **The Coase theorem says that if (a) property rights are well defined (or contracts enforced) and (b) transactions costs are negligible, then voluntary exchange can solve essentially all externality problems.**
- iv. **More over if (c) there are no significant income (original endowment) effects, then the final result tends to be the same regardless of the original assignment of property rights**
 - a. "a through c" are sometimes called the Coase theorem.
 - b. (It bears noting that part "c" of the "Coase theorem" requires the Pareto set to be composed of a single point, which is often the case in single dimensional diagrams, but not in multidimensional settings.)
- v. An Intuitive Example.
 - a. Suppose that a factory, Acme, uses a production process that produces smoke along with its marketable output. The wind mostly comes out of the West so that the smoke fall mostly on homeowners who live East of the factory .
 - b. The **weak form of the Coase theorem** (a and b) suggests that voluntary exchange can potentially solve this externality problem.
 - ♦ The home owners can band together and pay the firm to reduce its emissions either by reducing output or by using pollution control devices.
 - ♦ Gains to trade exist because at the margin, the firm realizes no profits from the last unit sold, but the home owners association is willing to pay a positive sum to get the firm to produce less.
 - ♦ Notice that very similar gains to trade would exist if the home owners initially had veto power over the firm's output. In this case, the firm would be willing to pay the home owner association for the privilege of producing its output and smoke.
 - c. **Whenever transactions costs are small**, contracts can be developed (trade can take place) that completely solve the externality problem in the sense that after the "Coasian contract" all gains from trade are realized, and net benefits are maximized.
- vi. The **strong form of the Coase theorem** holds if transactions costs are low and there are no important income effects that arise from the assignment of control over the resource or activity of interest.
 - a. In such cases, Coasian contracts will always reach the same output level, insofar as there is a unique output that maximizes social net benefits--as it often is in our diagrams.
 - b. In this case, the **final outcome is the same no matter who controls the resources** after all gains from trade are realized!

- c. (In other words, the gains to trade are exhausted at the same output level regardless of the initial assignment of control (property rights). For this and one other important insight about the nature of firms Ronald Coase won the Nobel Prize in economics.)

F. A Digression on Privatization

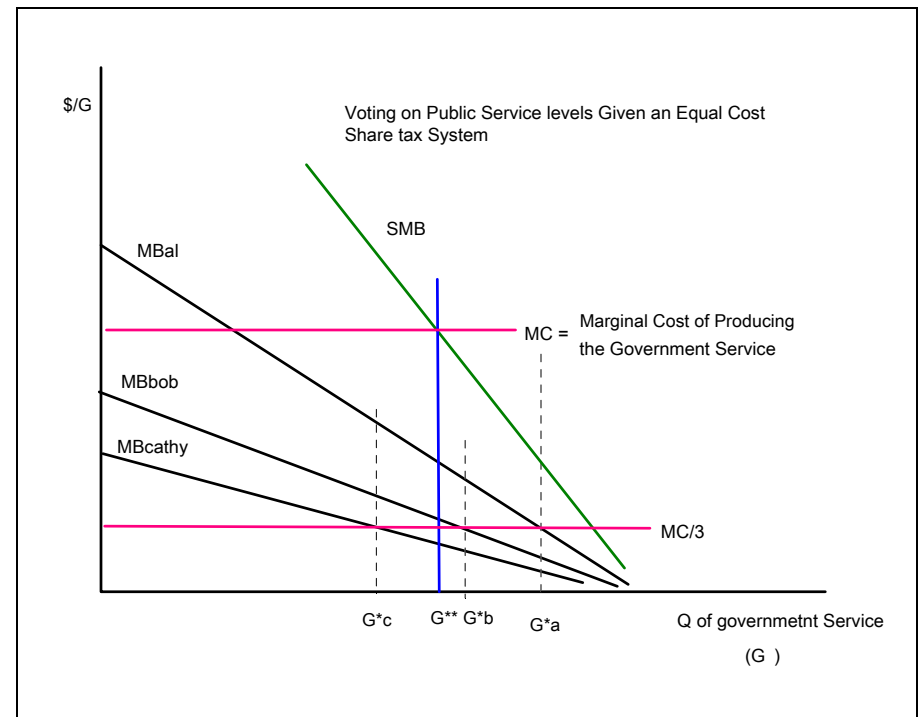
- i. In some cases, the reason for the externality is simply an improper specification of property rights.
 - ◆ For example, water use rights in dry areas are often undefined, with the result that as many people as want to can use as much water as they wish.
 - ◆ “Privatizing” water use rights (defining use rights and allowing them to be bought and sold) can solve this common property problem.
 - ◆ (The tragedy of the commons.)
- ii. For example, commons problems involving non-circulating or readily identifiable resources such as land, can be addressed by granting a person, firm, or club exclusive rights to control the usage of the resource in question.
 - ◆ (Privatization may solve such commons problem even if the "user rights" are not tradable, because *owners have no incentive to overuse their own resources.*)
 - ◆ However, not all externality problems are caused by communal property right systems.
 - ◆ There are also circulating/mobile economic resources that are difficult to assign rights to, as with the fish in the ocean, air, water, etc.

VIII. The Geometry of Voter Demands for Public Services and Regulations

- A. The net benefit maximizing model can also be used to characterize a voter's preferred level of a public service.'
- B. Most tax systems imply that a person's tax cost (burden) increases with an increase in services.
 - i. The increase in the tax burden associated with a change in a particular public service level is the marginal cost of the service for that voter.
 - ◆ Given that, **a voter will prefer the service level that equates his or her marginal benefit with his or her marginal tax cost for the service.**
 - ii. In most cases, the MB associated with a public service is higher for relatively rich persons than relatively poor persons, because most government services are normal goods. (as with education, roads, bicycle paths, national defense, etc.).
 - ◆ This implies that relatively rich persons have higher demands for public services than relatively poor persons, other things being equal.

- iii. However, tax prices often varies among voters--unlike market prices in competitive markets--because of the tax system.
 - ◆ For example, a progressive tax system implies that a rich voter pays a higher price than a poor voter for the same service.
 - ◆ Such price effects can cause a relatively wealth person to prefer less of a public service than a relatively poor person, even if the service is a normal good.
 - ◆ (Draw an example of such a case using MB and MTC curves.)

- C. A similar, but more indirect analysis of a voter's demand for a government service can be undertaken using utility functions, in which the tax system will affect the location and shape of the budget constraint facing the voters.
- D. The logic of a voter's demand for is basically similar, but in this case the regulations affect the marginal cost of goods and services purchased in other markets.



- i. Most regulations induce firms to use higher (money) cost forms of production.

- ii. Thus, regulations raise the marginal cost of goods produced by the affected industries.
- iii. In principle, the increase in the cost of other goods purchased by voters is the marginal cost of more stringent environmental or other regulations.
- iv. As in other NB maximizing choices, a voter will tend to prefer the level of regulatory stringency that sets the marginal benefit from more stringent rules
 - a. For examples pollution regulations and speed limits produce benefits (higher air or water quality, safe roads) but raise many kinds of production costs.
 - b. A voter's ideal regulation sets his marginal benefit from the regulation equal to his or her (indirect) marginal cost from more stringent rules.
- v. In other cases, nothing may be done, because transactions costs are too great. In such cases, it may cost more to solve the problem than is gained in social net benefits.

Part II: On the Mathematics of Neoclassical Public Economics

I. The Algebra and Calculus of Taxation and Revenues in a Partial Equilibrium Model

- A. All of the above can, of course, be done algebraically as well as geometrically.
 - i. One advantage of more mathematical approaches is that it is often possible to derive more general results.
 - ii. Another is that a mathematical approach may generate clearer predictions about relationships that can be estimated.
 - iii. Perhaps, surprisingly, in some cases a calculus-based analysis can be easier and generate clearer predictions than a geometric analysis can, because several relationships can be taken account of at the same time.
- B. As an exercise, consider the algebraic solution for prices and quantity in the case in which an excise tax is imposed on a single market, such as tires.
 - i. To simplify, assume that the demand and supply curves are linear, as often assumed when one estimates supply and demand curves.
 - ii. Let $Q_d = a - bP + cY$ and $Q_s = d + eP$
 - iii. Suppose a tax of t \$/unit is imposed on this market.
 - a. In equilibrium, $Q_d = Q_s$, but at two prices: P_c and P_s where $P_s = P_c - t$
 - b. (See the diagrams earlier in this lecture to appreciate this.)
 - c. This requires: $a - bP + cY = d + e(P-t)$ (where P is the consumer's price)
 - d. Solving for P requires gathering the P terms on one side and a bit of division.

- ◆ $a + cY - d + et = bP + eP = (b+e) P$
- ◆ which requires: $P = [a + cY + et] / [b+e]$
- ◆ Note that the consumer's price increases with t and with average consumer income.
- ◆ It falls as the slope of the demand curve increases (eg becomes more steeply downward sloping because b increases). In this case, more of the burden is shifted to supply.
- iv. The quantity purchased can be found by either substituting P into the demand curve or by substituting $P-t$ into the supply function.
 - ◆
 - ◆ $Q = a - b\{a + cY + et\} / [b+e] + cY$
 - ◆
- v. The tax revenue produced by an excise tax is tQ which is:
 - ◆
 - ◆ $T = t (a - b\{a + cY + et\} / [b+e] + cY)$
 - ◆
 - a. This is a bit messier than one would expect, but shows how the slopes of both the supply and demand curves, their intercepts, and other variables (here consumer income) affect the tax revenue generated.
 - b. Note that t now appears two times in the equation.
 - c. Total revenue is a quadratic function of tax rates.
- vi. This function is sometimes called the "Laffer curve" after a drawing supposedly made by Arthur Laffer on a napkin in a California restaurant, although many others have also had the same general idea.

C. Given a Laffer curve, one can derive the tax rate that maximizes total tax revenue.

- i. Differentiating $T = t (a - b\{a + cY + et\} / [b+e] + cY)$ with respect to t yields:
 - ◆
 - ◆ $dT/dt = (a - b\{a + cY + 2et\} / [b+e] + cY) = 0$ at t^*
 - ◆
- ii. A bit of algebra allows the solution to be characterized.
 - ◆
 - ◆ $a + cY + 2et\} / [b+e] = (a + cY) / b$
 - ◆ $2et\} / [b+e] = [(a + cY) / b] - a - cY$
 - ◆ $t^* = [(b+e)/2e] \{ [(a + cY) / b] - a - cY \}$
 - ◆

- ◆ which is linear in income by not in the slopes of the demand and supply curves.
- ◆
- iii. Of course, a simpler expression could have been derived for the case in which supply was horizontal $Q_s = P_s$ and for a demand curve without an income term. $Q_d = a - bP$
 - a. *Work that easier case out as an exercise.*
 - b. *You should get something like*
 - ◆ $t^* = (1+b) a/2b - a/2$ or
 - ◆ $t^* = a [(1+b) /2b - 1/2] = a/2b$
- iv. A tax higher than t^* generates less revenue than t^* .
 - a. There has been some debate about whether tax rates in the US are, or tend to be higher, than t^*
 - ◆ This was a claim made by many about US tax rates during the Reagan era.
 - ◆ In general national rates seem to be lower than t^* .
 - ◆ However, it is possible that taxes on mobile resources could be too high, as with taxes on capital.
 - ◆ (There is a bit of empirical work on this, but more could be done.)
 - ◆
- v. We will be introducing other mathematical tools as we go through the course.
 - ◆ The geometric tools reviewed in this handout are those upon which classic public economics is based.
 - ◆ These tools also provides the reference point and/or basis for a broad swath of contemporary public economics, as will be developed later in this course.

II. The Mathematics of Free Rider (Public Goods) Problems and Solutions

- A. Free-riding problems can be shown to occur in continuous representations of the production of pure public goods.
 - i. Suppose that G is a pure public good and X is a pure private good and that Al and Bob have similar utility functions and budget constraints, $U^A = u(G, X^A)$ and $W^A = G + PX^A$
 - ii. Al's utility maximizing quantity of the pure public good can be found by substituting for X^A in her utility function, $X^A = (W^A - G)/P$ and for G with $G = G^A + G^B$
 - ◆ Recall that if G is a pure public good, each person gain full benefits from the other's purchase of the good.

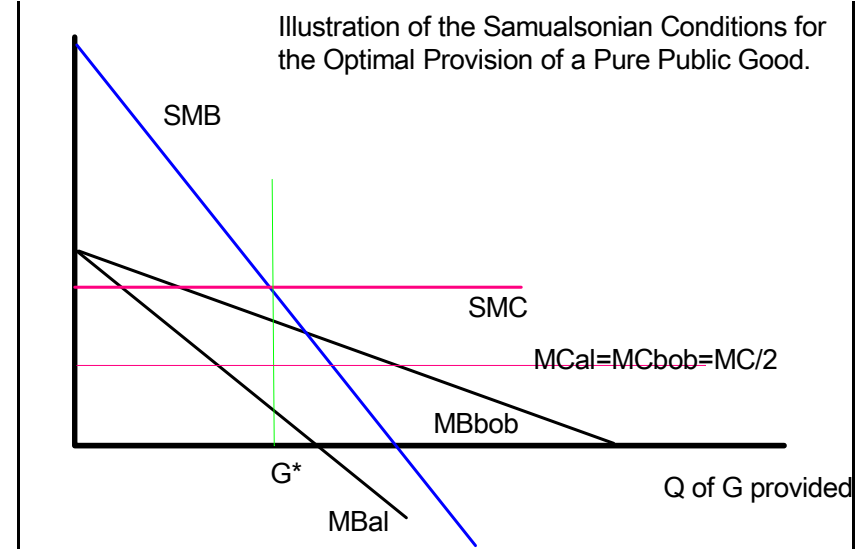
- ◆ Which yields: $U^A = u(G^A + G^B, (W^A - G)/P)$
- iii. Differentiating with respect to G yields Al's ideal purchase of G :
 - ◆ $U^A_G = u_G - u_X/P = 0$ at G^{A*}
 - a. Which implies that $G^{A*} = g(G^B, W^A, P)$
 - b. (This equation is A's demand for the public good, and also her "best reply function" for the free rider public goods game)
- iv. The mathematics for Bob yields a similar result:

$$G^{B*} = g(G^A, W^B, P)$$
- v. At the Nash equilibrium:

$$G^{B*} = g(G^{A*}, W^B, P)$$
 and $G^{B*} = g(G^{A*}, W^B, P)$
- vi. The **Pareto efficient level G** can be characterized using a Benthamite social welfare function (and other similar methods).
 - $W = U^A + U^B$
 - a. Differentiating with respect to G^A and G^B yields the first order conditions that describe the Pareto efficient levels of G^A and G^B
 - $W_{G^A} = u^A_G - u_X/P + u^B_G$
 - $W_{G^A} = u^B_G - u_X/P + u^A_G$
 - b. Note that **these two first order conditions describe functions that are "outside" of Al and Bob's best reply functions**, because they each require the external benefits of the other person to be taken account of.
 - ◆ [Draw the figure that illustrates this conclusion.]
 - ◆ Thus, the Nash equilibrium of this continuous version of the free rider game is Pareto sub optimal.
 - ◆ Too little of the public good is purchased by each.
 - However, some of the public good does get privately produced!
- B. There are a variety of solutions to public goods problem, including explicit coordination among those free riding, the formation of "public goods" clubs, private contracts (agreements) to contribute to produce the public good, and government action.
 - ◆ In small number cases like that of the 2x2, the persons affected may form a small club and perhaps hire a manager to solve the problem.
 - ◆ In large number settings it will of ten be cheaper to use the government that to form a new club for this purpose.

III. The Mathematics of the Pareto Optimal Provision of Pure Public Goods

- A. Within democracies, many government services can be understood as attempts to solve various free rider problems.
- i. In some cases, it will be easier for a group to take over production of a public good rather than to provide the proper Pigovian subsidies to encourage sufficient production.
 - ii. This may be done privately through clubs, or publicly through governments of one kind or another.
 - iii. In such ideal cases, **government can be thought of as a special kind of club with the power to tax.**
 - ◆ (Such clubs are good examples of what would be justified under contractarian theories of the state.)
- B. Samuelson in a classic 1954 paper on the optimal (utilitarian) supply and financing of a pure public good characterizes the ideal way to finance the "collective" production of such public goods.
- i. Ideally, the government would provide services at the Pareto optimal level, or, equivalently, at the level that maximizes social net benefits.
 - ii. The ideal taxes do not impose a deadweight loss. (Broad-based or lump sum taxes)
 - iii. They should raise just sufficient revenue to cover the cost of the public services.
 - iv. The sum of the marginal costs imposed on users of the public good should equal the marginal cost of producing them.
 - v. (These four conditions, optimal production of government services financed by an efficient tax system are sometimes called the **Samuelsonian conditions for the optimal provision of a pure public good.**)



- C. Samuelsonian's characterization of the Pareto optimal collective provision of a pure public good makes a number of assumptions about the characteristics of the goods in question and normative goals of policy makers, but these allow him to produce some interesting results.
- i. First we need some private choice notation: Let G be the level of a pure public good, let X_i be the level of a pure private good received by individual i , let $U_i = u(G, X_i)$ be the utility of individual i associated with a particular combination of the public good and private good received by i .
 - ii. Second, we need some macro-choice notation. Let W be a social welfare function and let $T(G, X) = 0$ be the technological frontier of combinations of the public good and private goods, with $X = \sum X_i$. Suppose there are N persons in the society of interest.
 - iii. The task of maximizing social welfare can be written as a Lagrangian:
 - ◆ $\max \mathcal{L} = W(U_1, U_2, U_3 \dots U_N) - \lambda (T(G, X))$
 - iv. Differentiating the Lagrangian with respect to $G, X_1, X_2, X_3 \dots X_N$, and λ yields the first order condition for the social welfare maximizing level of G and for the distribution of private goods--which we will ignore for the purposes of this derivation.
 - ◆ $\sum W_{U_i} U_{iG} = \lambda T_G$
 - ◆ $W_{U_i} U_{iX} = \lambda T_X$ for all $i = 1 \dots N$ (This represents N equations)

- ◆ $T(G, X) = 0$
- v. After obtaining the Lagrangians first order conditions, the next step is to manipulate the first order conditions into a form that is both economically interesting and useful. Samuelson uses a rather clever series of steps to do so.
 - a. First, divide the first and second condition to eliminate the lamda.
 - ◆ $[\sum W_{U_i} U_{iG}] / W_{U_j} U_{jX} = T_G / T_X$
 - ◆ (I have used the jth of the foc's for the private good to avoid confusion with "i" the counter for the summation in the numerator)
 - b. Since the denominator does not change with "i" it can be brought inside the brackets--because it is essentially a constant as far as this fraction is concerned.
 - ◆ $[\sum W_{U_i} U_{iG} / W_{U_j} U_{jX}] = T_G / T_X$
 - ◆ Now note that the foc's for the private goods imply that $W_{U_i} U_{iX} = W_{U_j} U_{jX}$
 - ◆ This condition holds for all "i" and "j" (for every person's private good).
 - c. This equivalence means that you can rewrite the equation under part b as:
 - ◆ $[\sum W_{U_i} U_{iG} / W_{U_i} U_{iX}] = T_G / T_X$
 - ◆ by substituting the various "i-terms" for the "j" term that we started with.
 - ◆ This allows us to simplify a bit:
 - ◆ $\sum [U_{iG} / U_{iX}] = T_G / T_X$
 - d. The ideal level of a pure public good will set the **sum of the marginal rates of substitution between the private and public good equal to the technological rate of transformation** between them.
 - ◆ (In the diagram above, this condition is represented by setting the sum of the marginal evaluation curves equal to the marginal cost of the pure public good.)
 - ◆ Note that **the optimal provision of a pure public good is completely independent of the social welfare function** used.
- D. There are a variety of practical problems with this characterization of the Pareto optimal provision of a pure public good, but the political one is of particular interest for the purposes of this course.
 - ◆ Most Pareto efficient provisions of a public good make individual tax payers "unhappy" with the amount of the public service provided, given their tax costs.

E. There is, however, a special case of the Samuelsonian Solution that avoids this problem, namely the **Lindahl tax** system.

- ◆ (Lindahl, a Swedish economist and student of Wicksell, surprisingly figured out this solution decades before Samuelson figured out how to characterize the Pareto efficient level of a pure public good..)

IV. Lindahl adds another condition to the three Samuelsonian conditions for Pareto efficient provision of a pure public good.

- ◆ **Lindahl** suggests that the *taxes used to finance public services should equate marginal benefits and marginal costs for individuals at the desired output of government services.*
- ◆ Lindahl taxes are, thus, said to be idealized **benefit taxes**.
- i. They can also be applied to finance the Pareto efficient level of a pure public service, G^* , characterized by Samuelson..
- ii. **Under Lindahl taxation, everyone in the society of interest is prefers the Pareto optimal level of public goods to all others.**
 - ◆ [See the in-class lecture notes for an illustration of Lindahl taxation.]
 - a. Note that under a perfect benefit tax of this sort, each person "demands" the same output of the pure public good, namely G^{**} .
 - b. This contrasts with the less restricted Samuelsonian case, in which persons are very likely to disagree about the best service level to provide!
 - ◆ (Consider for example the special case in which the cost of the service is shared equally among three persons with different marginal benefit curves.)
 - ◆ Those whose marginal tax cost are below their marginal benefits from the service will demand more, whereas those whose marginal tax cost is above their marginal benefits will want less!
 - c. Lindahl taxes would induce unanimous agreement about the level of a pure public good to be provided, and so suggest that collective action to solve public goods problems in this way may well be undertaken.
 - ◆ [This would, of course, require providing institutions that assure that taxes are paid. Free riding on contributions to the public good remain rational even under Lindahl taxes.]

V. Lindahl Taxation if resident-citizens have Cobb-Douglas utility functions.

- A. There is an interesting special case of Lindahl taxation that arises when all persons on the polity have identical tastes that can be represented with a Cobb-Douglas utility function, but they have different wealth endowments.
- i. Let G be the pure public good and C be private consumption and assume that “ t ” is the price of the pure public good and “ P ” is the price of the pure private good.
 - ii. In this case, “Mr/Ms i ” maximizes $U_i = C_i^a G^{1-a}$ subject to $W_i = tG + PC_i$
- B. Mr/Ms i ’s demand for the government services given this tax-price system for financing the public service can be derived by (i) forming a Lagrangian (which works well for C-D functions), (ii) differentiating with respect to C_i , G , and λ , and (iii) doing some clever algebra.
- i. $L = C_i^a G^{1-a} + \lambda (W_i - tG - PC_i)$
 - ii. Differentiating with respect to C_i , G , and λ , setting the results equal to zero, and some simple algebra yields:
 - a. $(1-a)C_i^a G^{-a} = \lambda t$
 - b. $aC_i^{a-1} G^{1-a} = \lambda P$
 - c. $W_i = tG - PC_i$
 - iii. Some more “easy” algebra on these three first order conditions allows Mr/Ms i ’s demand for G to be characterized.
 - a. Dividing “a” by “b” yields: $[(1-a)/a] [C_i^*/G^*] = t/P$
 - b. Solving for C_i^* yields: $C_i^* = G^* (t/P) [a/(1-a)]$
 - c. Substituting for C in the budget constraint: $W = tG^* - PC_i^*$ yields $W_i = tG^* - P G^* (t/P) [a/(1-a)]$
 - d. Simplifyng and solving for G_i^* yields $G_i^* = (1-a)W_i/t$
 - ◆ As normally the case with C-D utility functions, each person spends a **particular fraction of their wealth** on the goods of interest, here $(1-a)$, and the amount purchased varies with the price, here (t) .
 - ◆ In the special case of interest here, everyone spends the same fraction of their wealth on each type of good.
 - iv. A Lindahl tax system has the property that it induces each person to demand the same quantity of the public good. (See the diagrams above.)
 - a. We can use this property to characterize the Lindahl tax prices for this polity.

- b. Each person pays a different price under a Lindahl system, here t
 - c. Those taxes induce each to purchase the same quantity of goods so for persons “ i ” and “ j ,” $G_i^* = G_j^*$ at their respective Lindahl taxes.
 - d. Assume that G^{**} is the Pareto optimal quantity of the pure public good and that “ t ” satisfies the Samuelsonian condition and, so, is just sufficient to pay for the pure public good [that is: $\sum t_i G^{**} = c(G^{**})$].
 - ◆ In this case the Lindahl taxes satisfy: $(1-a)W_i/t_i = (1-a)W_j/t_j$
 - ◆ and $t_i/t_j = W_i/W_j$
 - ◆ That is to say if “ i ” has twice as much wealth as “ j ”, i ’s tax price should be twice as high as j ’s.
 - e. **In this special case [identical Cobb-Douglas tastes and $\sum t_i G^{**} = c(G^{**})$], a progressive tax that has marginal tax rates equal to relative wealth (or income) is a Lindahl tax system.**
- C. Of course, preferences differ and are not likely to all take the form of Cobb-Douglas utility functions, and this tax financing scheme might not be sufficient to pay for the pure public service.
- i. However, it is interesting to note that Lindahl taxation would require progressive wealth (or income) taxation, **except for** differences in tastes and economies and/or diseconomies of scale in producing government services.
 - ii. Moreover, it is interesting to note that in “easy circumstances” one does not have to determine each person’s unique marginal benefits schedule (or marginal rate of substitution).
 - ◆ Are there any communities that have progressive real estate taxes, or are they all flat systems? If so, is there more consensus about the level of services in such communities?
 - ◆ If there is not greater consensus, this would suggest that taste differences were important. People of similar income may have quite different demands for bicycle paths, day care centers, parks etc.. Explain.

Appendix: On the Simple Calculus of Demand and Supply using the Substitution Method

A. In some cases, it is possible to "substitute" the constraint(s) into an objective function to create a new "composite function" that fully represents the effects of the constraint on the original objective function.

- i. Generally, the substitution method attempts to reduce the number of control variables that have to be taken account of.
- ii. The substitution method generally entails the following steps:
 - a. First., use the constraints to completely specify of the control variables as functions of the subset of control variables that are of most interest.
 - b. Second, substitute these relationships into the objective function to form a new objective function (that reflects all the constraints).
 - c. Third, differentiate with respect to each of the control variables that remain. (Often this will be just a single variable.)
 - d. If the new objective function is strictly concave, the optimal value(s) of the control variable(s) given the constraint(s) is the one(s) that sets the first derivative equal to zero.
- iii. For example consider the separable utility function: $U = x^5 + y^5$ to be maximized subject to the budget constraint $100 = 10x + 5y$. (Good x costs 10 \$/unit and good y costs 5 \$/unit. The consumer has 100 dollars to spend.)
- iv. We can rewrite the constraint as $y = [100 - 10x]/5 = 20 - 2x$
- v. Substituting this for Y in the objective function (here, the utility function) yields a new function *entirely in terms of x*: $U = x^5 + (20 - 2x)^5$
- vi. This new function accounts for the fact that every time one purchases a unit of x one has to reduce his consumption of y. (Why?)
- vii. Note also that this new objective function has just one control variable, x.
- viii. Differentiating with respect to x and setting the result equal to zero allows the utility maximizing quantity of x to be characterized:

$$d[x^5 + (20 - 2x)^5] / dx = .5 x^{-.5} + .5(20-2x)^{-.5}(-2) = 0 \text{ (at U max)}$$
 - a. The derivative will have the value zero at the constrained utility maximum. Setting the above expression equal to zero, moving the second term to the right, then squaring and solving for x yields:

$$4x = 20 - 2x \Rightarrow 6x = 20 \Rightarrow x^* = 3.33$$
 - b. Substituting x^* back into the budget constraint yields a value for y^*

$$y = 20 - 2(3.33) \Rightarrow y^* = 13.33$$
- ix. No other point on the budget constraint can generate a higher utility level than that $(x^*, y^*) = (3.33, 13.33)$.

B. More Illustrations of the substitution method.

- i. Derivation of Demand Curve, general case for two-good world
 - a. Suppose that Al's utility function is $U = u(A, B)$, with positive first derivatives, negative second derivatives, and positive or zero second derivatives.
 - b. Suppose also that she has W dollars to spend on these goods, whose prices are P^A and P^B .
 - c. The latter implies that Al's budget constraint is $W = AP^A + BP^B$.
 - d. Note that as long as "more as better" (positive first derivatives), she will be choose a combination of goods along her budget constraint, rather than inside it. Note also that is she purchase A units of the second good, she will necessarily purchase $B = [W - AP^A] / P^B$ units of the second good.
 - e. This relationship can be substituted into Al's utility function to create a "new" objective function that takes account of the budget constraint: $U = u(A, [W - AP^A] / P^B)$
 - f. Differentiating this function with respect to A and setting the result equal to zero yields a first order condition that describes Al's purchase of good A.
 - g. $U_A - U_B (P^A / P^B) = 0$ where U_A denotes the partial derivative of U with respect to A and U_B denotes the partial derivative with respect to B.
 - h. Note that this first order condition has a clear meaning. The first term, U_A , is the marginal utility of good A. The second term, $- U_B (P^A / P^B)$, is the (subjective) marginal opportunity cost of consuming A in terms of lost utility from good B.
 - i. $U_A - U_B (P^A / P^B) = 0$ implies that at A^* , $U_A = U_B (P^A / P^B)$, which means that A will be consumed at the level where marginal benefit (marginal utility) equal's marginal cost (marginal opportunity cost).
 - j. The implicit function theorem--about which we will have more to say next lecture--allows A^* to be characterized as a function of the other variables in the first order condition: $A^* = a(P^A, P^B, W)$.
 - k. $A^* = a(P^A, P^B, W)$ is Al's demand function for good A.
 - l. Al's demand function for good B can be solved by substituting this function into $B = [W - AP^A] / P^B$, so $B^* = [W - A^*P^A] / P^B$.
- ii. A monopolist's profit maximizing output level can be characterized in a similar manner.
 - a. Suppose that the inverse demand function facing a monopolist is $P = d(Q, Y)$, with inverse demand (price) falling with increases in Q and increasing with increases in Y (consumer income).
 - b. (An inverse demand function maps Qs into Ps rather than Ps into Qs.)
 - c. The monopolist's profit, Π , is his total revenue, R, less his total cost C.

- d. Suppose that his total costs are an increasing function of output $C = c(Q, w)$ and with wage rates.
- e. Substituting for price and cost we obtain the monopolist's profit function:
- f. $\Pi = R - C = Q d(Q, Y) - c(Q, w)$
- g. Differentiating with respect to Q and setting the result equal to zero generates a first order condition that characterizes the firm's profit maximizing output.
- h. $\Pi_Q = R_Q - C_Q = d(Q, Y) + Q d_Q - c_Q = 0$ at Q^*
- i. Or, doing a bit of subtraction, Q^* is such that $d(Q, Y) + Q d_Q = c_Q$
- j. Again there is a nice economic interpretation of the first order condition. The first term, $d(Q, Y) + Q d_Q$, is marginal revenue, and the second, c_Q , is marginal cost.
- k. So the profit maximizing firm will produce at the output that sets marginal revenue equal to marginal cost.
- l. The implicit function theorem--about which we will talk more in the next lecture--allows Q^* to be written as a function of the other variables that are in the first order condition: $Q^* = q(w, Y)$.
- m. The price at which the output is sold can be found by substituting Q^* into the inverse demand function: $P^* = d(Q^*, w)$, and monopoly profit by substitution this and Q^* into the original profit function: $\Pi^* = Q^* d(Q^*, Y) - c(Q^*, w)$.
- iii. The substitution method can also be used, as in the first illustration, in cases in which the various relationships (functions) are assumed to take specific forms, as with the Cobb-Douglas, or CES, etc...
- iv. In general, the substitution method is a quite powerful technique for developing general models in settings in which a few key variables and constraints are focused on.

Appendix: Comparative Statics using the Implicit Function Theorems

- A. It bears noting, however, that there are many cases in which there will not be nice closed form solutions for the first order conditions or for the market phenomena of interest, even given specific functional forms for the relationships of interest.
 - i. Moreover, also many cases in which we may not know the functional form of the relationships of interest, or want to make specific assumptions about functional forms.
 - In many cases, it will be EASIER to develop models with abstract functional representations of the relationships of interest.
 - **In such cases, comparative statics, surprisingly, can still be done!**
 - ii. But, in order to do so, we require two additional tools beyond ordinary calculus and algebra: namely the implicit function theorem and the implicit function differentiation rule.

- B. The implicit function theorem allows a variety of properties to be deduced from the first order conditions, including many that are useful for comparative statics.
 - i. The **implicit function theorem** implies that the first order conditions to be used: to characterize the solution (optimal value of the control variable(s)) as a function of the parameters of the optimization problem.
 - ii. The **implicit function differentiation** rule allows the comparative statics of such functions to be derived.
 - a. That is to say, the "implicit function differentiation rule" can be used to describe how the "ideal values" of the control variables change as parameters of the choice problem change.
 - b. For example, in a consumer choice problem one can determine rate at which the utility maximizing quantity of a good changes as the consumer's income changes or as some price changes. ($dQ^*/dY, dQ^*/dPs, \dots$) *The latter is the slope of the consumer's demand curve.*

C. The Implicit Function Theorem (see Chiang 205 - 206, La Fuente 5.2):

Given a function such that: $F(Y, X_1, X_2, \dots, X_m) = 0$

- i. Where the function F has continuous partial derivatives $F_Y, F_{X_1}, F_{X_2}, \dots, F_{X_m}$,
 - ii. and at point $(Y^o, X_1^o, X_2^o, X_3^o, \dots, X_m^o)$ satisfying condition i, F_Y is nonzero,
 - iii. *Then* there exists an m -dimensional neighborhood, N , of the point $(Y^o, X_1^o, X_2^o, X_3^o, \dots, X_m^o)$,
 - iv. in which an implicit function exists that characterizes each of the "ideal" values of the control variables of F as a function of non-control variables (the "parameters" of the choice problem): $Y = f(X_1, X_2, X_3, \dots, X_m)$
 - v. This function satisfies $Y^o = f(X_1^o, X_2^o, X_3^o, \dots, X_m^o)$ in particular, and more generally, $Y = f(X_1, X_2, X_3, \dots, X_m)$ for all points within the neighborhood.
 - vi. This gives the function the status of an identity within neighborhood N .
 - vii. Moreover, the implicit function, f , is continuous and has continuous partial derivatives with respect to X_1, X_2, \dots, X_m .
 - a. In cases where there is a single first order condition, there is a fairly straightforward method by which these partial derivative can be computed.
 - b. (See the implicit function differentiation rule below.)
- D. The N -equation version of the implicit function theorem is broader in scope but essentially similar. (See Chiang 210 - 211, la Fuente 5.2.)

E. In microeconomic applications, the "zero function," $F(Y, X_1, X_2, \dots, X_m) = 0$, is usually the first order condition of some optimization problem, and the implicit function is the individual's (or firm's) demand (or supply) function. However, the theorem applies to ANY function that equals zero.

F. The **implicit function differentiation rule** is used to characterize the partial derivatives of the implicit function "generated" as above.

i. The rule is surprisingly simple (la Fuente, theorem 2.1).

ii. The partial derivative of implicit function Y with respect to X_i is simply:

$$Y_{X_i} = F_{X_i} / -F_Y \quad (\text{where subscripts denote partial derivatives with respect to the variable subscripted}).$$

iii. The **derivation** of this rule relies on the total derivatives of F .

a. Recall that $F(Y^o, X_1^o, X_2^o, X_3^o, \dots, X_m^o) = 0$

b. Thus the total derivative of F have to add up to zero.

$$dY F_Y + dX_1 F_{X_1} + \dots + dX_m F_{X_m} = 0$$

c. Consequently, if we allow only X_i and Y to vary,

$$dY F_Y + dX_i F_{X_i} = 0$$

d. Solving this expression for dY/dX_i yields:

$$dY/dX_i = F_{X_i} / -F_Y$$

iv. The implicit function differentiation rule allows one to characterize how the solution to an optimization problem varies as parameters of the problem vary.

G. Example: **Properties of an individual's "abstract" demand function**

i. Suppose that "Al" has a utility function, $U = u(X_1, X_2)$ which is monotone increasing in X_1 and X_2 , twice differentiable and strictly concave. The latter may be assured by assuming that: $U_{X_1 X_2} \geq 0$, $U_{X_1 X_1} < 0$, and $U_{X_2 X_2} < 0$.

ii. Al wants to find the utility maximizing combination of X_1 and X_2 given the budget constraint that he faces, $W = P_1 X_1 + P_2 X_2$.

iii. Using the substitution method: $U = u(X_1, (W - P_1 X_1)/P_2)$

iv. Differentiating with respect to X_1 yields: $U_{X_1} + U_{X_2} (-P_1/P_2) = 0$

a. The value of X_1 that satisfies this first order condition will maximize utility.

b. Denote such that value of X_1 as X_1^*

v. Note that at X_1^* , the first order condition is a function like F in the definition of the implicit function theorem; that is to say, the "foc" always equals zero at X_1^* . Since the first order condition is differentiable (remember that we assumed that U was twice differentiable), an implicit function exists that characterizes X_1^* as a function of the other parameters of the choice problem.

$$X_1^* = x(W, P_1, P_2)$$

vi. Economists refer to this function as Al's **demand function** for X_1 .

vii. The effect of a change in the price of good 1, P_1 , on Al's demand for good 1 can be characterized using the implicit function differentiation rule:

a. $X_1^*_{P_1} = F_{P_1} / -F_{X_1}$

b. Given our first order condition in equation iv above, $F_{P_1} / -F_{X_1}$ can be written as:

$$[U_{X_1 X_2} (-X_1/P_2) + U_{X_2} (-1/P_2) - U_{X_2 X_2} (P_1/P_2)(-X_1/P_2)]$$

$$-[U_{X_1 X_1} + 2 U_{X_1 X_2} (-P_1/P_2) + U_{X_2 X_2} (-P_1/P_2)^2]$$

viii. This expression is determined by carefully calculating the derivatives of the foc, $U_{X_1} + U_{X_2} (-P_1/P_2)$, with respect to P_1 and X_1 .

H. The "sign" of this derivative of our implicit function tells us whether Al's demand curve slopes downward or not.

i. The "sign" is jointly determined by all the partial derivatives in the expression above.

ii. Most of these have already been characterized by our assumptions about Al's utility function and his budget constraint.

a. From the original characterization of U we know that all of the first partial derivatives are positive

b. We also know that all of the second derivatives are negative (This implies that both X_1 and X_2 are goods that exhibit diminishing marginal utility.).

c. We also know that the cross partial is positive (an increase in good 2 increases the marginal utility of good 1).

d. Together these characteristics of U imply that Al's demand curve is **downward sloping**, $X_1^*_{P_1} < 0$.

e. (As an exercise, compute the derivative of the demand function with respect to Al's wealth and see whether "positive cross partials" also rule out inferior goods.)

- I. The implicit function approach has a wide range of applications in economic models and in game theory.
- ◆
 - (There are also multi-equation version of the theorem and differentiation rule, as developed in Chaing 8.5. These generalizations, however, are not very widely used.)